ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{(w_a+w_b+w_c)(m_a+m_b+m_c)}{(h_a+h_b+h_c)(r_a+r_b+r_c)} \le \frac{(a+b+c)^2}{3(ab+bc+ca)} \le \frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a = \sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\le} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}$$

$$\overset{A-G}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}} = \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}} = \sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$= \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \Rightarrow \sum_{cyc} w_a \le \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)}$$

$$\therefore \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)} \overset{Chu-Yang}{\ge} \frac{2R(4R+r).\sqrt{4s^2-28Rr+29r^2}}{\sqrt{(\sum_{cyc} h_a)(\sum_{cyc} r_a)}.\sum_{cyc} ab} \overset{?}{\ge} \frac{\sum_{cyc} a^2}{\sum_{cyc} ab}$$

$$\Leftrightarrow 2R^3(4R+r)(4s^2-28Rr+29r^2) \ge (s^2+4Rr+r^2)(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow -s^6+(4Rr+r^2)s^4+(32R^4+8R^3r+16R^2r^2+8Rr^3+r^4)s^2$$

$$-r(224R^5-176R^4r+6R^3r^2+48R^2r^3+12Rr^4+r^5) \overset{(1)}{\ge} 0$$

$$\text{Now, via Gerretsen, } P = -s^4(s^2-4R^2-4Rr-3r^2)$$

$$-(4R^2+2r^2)s^2(s^2-4R^2-4Rr-3r^2)$$

$$-(16R^4-8R^3r-4R^2r^2-5r^4)(s^2-16Rr+5r^2) \ge 0$$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\ge P$

$$\Leftrightarrow 16t^5-16t^4-15t^3-14t^2-46t+12 \ge 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(16t^4+16t^3+17t^2+20t-6) \ge 0 \to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{(r_a+r_b+r_c)(m_a+m_b+m_c)}{(w_a+w_b+w_c)(h_a+h_b+h_c)} \to (i)$$

$$\text{We shall now evaluate : } \sum_{cyc}\frac{4s(s-a)}{(b+c)^2} \text{ and it's } = \sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$= 3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$= 4s\sum_{cyc}\frac{1}{b+c} - 4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2 - \frac{2}{2s(s^2+2Rr+r^2)}.\sum_{cyc}(b+c)\right)$$

$$= \frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - \frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2} + \frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc} \frac{4s(s-a)}{(b+c)^2} \overset{(*)}{=} \frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R+r)}{(s^2 + 2Rr + r^2)^2}$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ *and* $s^2 - (m+n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ and $n = 2(R-2r).\sqrt{R^2 - 2Rr}$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(**)}{\leq} 0$$

$$\text{Now, } w_a + w_b + w_c = \frac{\sqrt{bc}}{b+c}.\sqrt{4s(s-a)} \overset{CBS}{\leq} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc} \frac{4s(s-a)}{(b+c)^2}} \overset{\text{via }(*)}{=}$$

$$\sqrt{(s^2 + 4Rr + r^2).\frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R+r)}{(s^2 + 2Rr + r^2)^2}} \overset{?}{\leq} \sqrt{s^2 + 21Rr + 12r^2}$$

$$\Leftrightarrow (R - 5r)s^4 + r(8R^2 - 2Rr + 6r^2)s^2 +$$

$$r^2(84R^3 + 116R^2r + 61Rr^2 + 11r^3) \underset{(2)}{\overset{?}{\geq}} 0 \text{ and it's trivially true if : } R - 5r \geq 0$$

and so, we now focus on the *case* when : $R - 5r < 0$ *and* then, via $(**)$,

$$(R - 5r)\left(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3\right) \geq 0$$

$\therefore$ in order to prove (2), it suffices to prove :

$$\text{LHS of (2)} \geq (R - 5r)\left(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3\right)$$

$$\Leftrightarrow (R^3 + 2R^2r - 26Rr^2 + 4r^3)s^2 \underset{(3)}{\overset{?}{\geq}} r(16R^4 - 89R^3r - 86R^2r^2 - 30Rr^3 - 4r^4)$$

Case 1 $R^3 + 2R^2r - 26Rr^2 + 4r^3 \geq 0$ *and* then : LHS of (3) $\overset{Gerretsen}{\geq}$

$$(R^3 + 2R^2r - 26Rr^2 + 4r^3)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of (3)}$$

$$\Leftrightarrow 29t^3 - 85t^2 + 56t - 4 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(29t^2 - 27t + 2) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow$ (3) is true

Case 2 $R^3 + 2R^2r - 26Rr^2 + 4r^3 < 0$ *and* then : LHS of (3) $\overset{Gerretsen}{\geq}$

$$(R^3 + 2R^2r - 26Rr^2 + 4r^3)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of (3)}$$

$$\Leftrightarrow t^5 - t^4 - t^3 + t^2 - 8t + 4 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(t^4 + t^3 + t^2 + 3t - 2) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$\because t \overset{Euler}{\geq} 2 \Rightarrow$ (3) is true $\therefore$ combining both *cases*, (3) $\Rightarrow$ (2) is true $\forall \Delta ABC$

$$\therefore w_a + w_b + w_c \leq \sqrt{s^2 + 21Rr + 12r^2} \Rightarrow \frac{(w_a + w_b + w_c)(m_a + m_b + m_c)}{(h_a + h_b + h_c)(r_a + r_b + r_c)}$$

$$\overset{Chu-Yang}{\leq} \frac{2R.\sqrt{(s^2 + 21Rr + 12r^2)(4s^2 - 16Rr + 5r^2)}}{(s^2 + 4Rr + r^2)(4R+r)} \overset{?}{\leq} \frac{(a+b+c)^2}{3(ab+bc+ca)}$$

$$= \frac{4s^2}{3(s^2 + 4Rr + r^2)} \Leftrightarrow 4(4R+r)^2s^4 \overset{?}{\geq} 9R^2(s^2 + 21Rr + 12r^2)(4s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow (28R^2 + 32Rr + 4r^2)s^4 - R^2r(612R + 477r)s^2$$

$$+R^2r^2(3024R^2 + 783Rr - 540r^2) \underset{(4)}{\overset{?}{\geq}} 0$$

$$\text{Now, via Gerretsen, } Q = (28R^2 + 32Rr + 4r^2)(s^2 - 16Rr + 5r^2)^2$$

$$+r(284R^3 + 267R^2r - 192Rr^2 - 40r^3)(s^2 - 16Rr + 5r^2) \geq 0$$

$\therefore$ in order to prove ④, it suffices to prove : LHS of ④ $\geq Q$

$$\Leftrightarrow 400t^4 - 77t^3 - 1551t^2 + 160t + 100 \geq 0$$

$$\Leftrightarrow (t-2)\left(400t^3 + 657t^2 + 53t(t-2) + 13\left(t^2-4\right) + t + 2\right) \geq 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ④ is true $\therefore \dfrac{(w_a + w_b + w_c)(m_a + m_b + m_c)}{(h_a + h_b + h_c)(r_a + r_b + r_c)} \leq$

$\dfrac{(a+b+c)^2}{3(ab+bc+ca)} \rightarrow$ (ii) $\therefore$ (i) and (ii) combined with $3\sum\limits_{cyc} a^2 \geq \left(\sum\limits_{cyc} a\right)^2$ gives :

$$\frac{(w_a + w_b + w_c)(m_a + m_b + m_c)}{(h_a + h_b + h_c)(r_a + r_b + r_c)} \leq \frac{(a+b+c)^2}{3(ab+bc+ca)} \leq \frac{a^2+b^2+c^2}{ab+bc+ca}$$

$\leq \dfrac{(r_a + r_b + r_c)(m_a + m_b + m_c)}{(w_a + w_b + w_c)(h_a + h_b + h_c)}$ $\forall$ ABC, with equality iff Δ ABC is equilateral (QED)