ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$w_a + w_b + w_c \le h_a + h_b + h_c + \frac{38}{25}(R - 2r)$$

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$$\sum_{cyc} \frac{4s(s-a)}{(b+c)^2} = \sum_{cyc} \frac{(b+c)^2 - a^2}{(b+c)^2} = 3 - \sum_{cyc} \frac{(2s - (2s-a)^2)}{(2s-a)^2}$$

$$= 4s\sum_{cyc} \frac{1}{b+c} - 4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2 - \frac{2}{2s(s^2+2Rr+r^2)}.\sum_{cyc}(b+c)\right)$$

$$= \frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - \frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2} + \frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc} \frac{4s(s-a)}{(b+c)^2} \overset{(*)}{=} \frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

Now, Rouche $\Rightarrow s^2 - (m-n) \ge 0$ and $s^2 - (m+n) \le 0$, where $m = 2R^2 + 10Rr - r^2$ and $n = 2(R-2r).\sqrt{R^2 - 2Rr}$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \le 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(**)}{\le} 0$$

$$\text{Now, } w_a + w_b + w_c = \frac{\sqrt{bc}}{b+c}.\sqrt{4s(s-a)} \overset{CBS}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}} \overset{via\ (*)}{=}$$

$$\sqrt{(s^2+4Rr+r^2).\frac{s^4 + (20Rr+18r^2)s^2 + r^3(4R+r)}{(s^2+2Rr+r^2)^2}} \overset{?}{\le} \sqrt{s^2 + 21Rr + 12r^2}$$

$$\Leftrightarrow (R - 5r)s^4 + r(8R^2 - 2Rr + 6r^2)s^2 +$$

$$r^2(84R^3 + 116R^2r + 61Rr^2 + 11r^3) \underset{①}{\overset{?}{\ge}} 0 \text{ and it's trivially true if : } R - 5r \ge 0$$

and so, we now focus on the case when : $R - 5r < 0$ and then, via $(**)$,

$$(R-5r)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3) \ge 0$$

$\therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \ge (R-5r)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3)$$

$$\Leftrightarrow (R^3 + 2R^2r - 26Rr^2 + 4r^3)s^2 \underset{②}{\overset{?}{\ge}} r(16R^4 - 89R^3r - 86R^2r^2 - 30Rr^3 - 4r^4)$$

Case 1 $R^3 + 2R^2r - 26Rr^2 + 4r^3 \ge 0$ and then : LHS of ② $\overset{Gerretsen}{\ge}$

$$(R^3 + 2R^2r - 26Rr^2 + 4r^3)(16Rr - 5r^2) \overset{?}{\ge} \text{RHS of ②}$$

$$\Leftrightarrow 29t^3 - 85t^2 + 56t - 4 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(29t^2 - 27t + 2) \overset{?}{\ge} 0 \to \text{true} \because t \overset{Euler}{\ge} 2$$

$$\Rightarrow \textcircled{2} \text{ is true}$$

Case 2 $R^3+2R^2r-26Rr^2+4r^3<0$ *and then* : LHS of $\textcircled{2} \overset{\text{Gerretsen}}{\geq}$

$$(R^3+2R^2r-26Rr^2+4r^3)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of } \textcircled{2}$$

$$\Leftrightarrow t^5-t^4-t^3+t^2-8t+4 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(t^4+t^3+t^2+3t-2) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{2} \text{ is true} \therefore \text{combining both cases}, \textcircled{2} \Rightarrow \textcircled{1} \text{ is true } \forall \Delta ABC$$

$$\therefore w_a+w_b+w_c \leq \sqrt{s^2+21Rr+12r^2} \overset{?}{\leq} h_a+h_b+h_c+\frac{38}{25}(R-2r)$$

$$\Leftrightarrow s^2+21Rr+12r^2 \overset{?}{\leq} \frac{(s^2+4Rr+r^2)^2}{4R^2}+\frac{1444}{625}(R-2r)^2$$

$$+\left(\frac{s^2+4Rr+r^2}{R}\right).\frac{38}{25}(R-2r)$$

$$\Leftrightarrow 625(s^2+4Rr+r^2)^2+5776R^2(R-2r)^2+3800R(R-2r)(s^2+4Rr+r^2)$$

$$-2500R^2(s^2+21Rr+12r^2) \underset{\circledast}{\overset{?}{\geq}} 0 \text{ and } \because 625(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } \circledast, \text{it suffices to prove : LHS of } \circledast \overset{?}{\geq} 625(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (325R^2+4350Rr-1250r^2)s^2+1444R^4-15101R^3r-45874R^2r^2$$

$$+24350Rr^3-3750r^4 \underset{\circledast\circledast}{\overset{?}{\geq}} 0$$

$$\text{We have : LHS of } \circledast\circledast \overset{\text{Rouche}}{\geq}$$

$$(325R^2+4350Rr-1250r^2)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$+1444R^4-15101R^3r-45874R^2r^2+24350Rr^3-3750r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2094R^3+1037R^2r-3125Rr^2+1250r^3) \underset{\circledast\circledast\circledast}{\overset{?}{\geq}}$$

$$2(R-2r).\sqrt{R^2-2Rr}.(325R^2+4350Rr-1250r^2) \text{ and } \because R-2r \overset{\text{Euler}}{\geq} 0$$

$$\therefore \text{in order to prove } \circledast\circledast\circledast, \text{it suffices to prove :}$$

$$(2094R^3+1037R^2r-3125Rr^2+1250r^3)^2 \overset{?}{>}$$

$$4(R^2-2Rr)(325R^2+4350Rr-1250r^2)^2$$

$$\Leftrightarrow 3962336t^6-6122044t^5-61832131t^4+187133750t^3-80891875t^2$$

$$+4687500t+1562500 \overset{?}{>} 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)\big((t-2).T+81568702\big)+19136601\right)+219101800\right)$$

$$+252810000 \overset{?}{>} 0 \text{ (with } T=3962336t^2+25576644t+47684957)$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \circledast\circledast\circledast \Rightarrow \circledast\circledast \Rightarrow \circledast \text{ is true} \Rightarrow w_a+w_b+w_c \leq$$

$$h_a+h_b+h_c+\frac{38}{25}(R-2r) \;\forall\, ABC, \text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$