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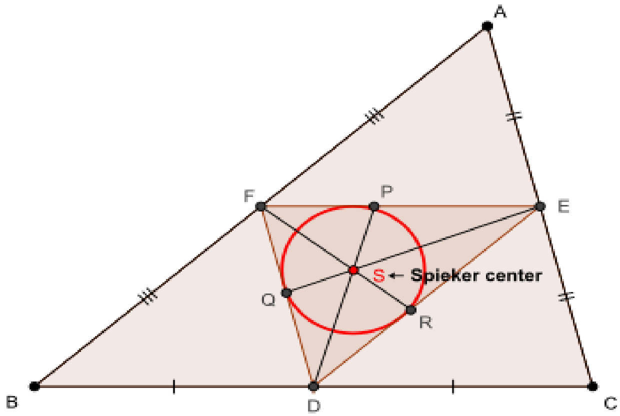
In any Δ ABC, the following relationship holds :

$$\frac{1}{h_a}+\frac{1}{w_a}+\frac{1}{m_a}\le\left(\frac{5R}{3r}-\frac{1}{3}\right)\frac{1}{p_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of }\Delta DEF,\therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= \mathrm{Rr}\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2\mathrm{Rr}\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$= \frac{\mathrm{Rr}}{8\mathrm{Rrs}}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s}-2\mathrm{Rr}$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s}+2\mathrm{Rr}$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right) = \frac{ab+ca}{4}-2\mathrm{Rr} \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**) \Rightarrow 2\mathrm{AS}^2 = \frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2\mathrm{AS}^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\,\mathrm{AFS}, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{\mathrm{AS}}{\cos\frac{A-B}{2}} = \frac{c\mathrm{AS}}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2\mathrm{AS}} \text{ and via sine law on } \Delta\,\mathrm{AES}, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2\mathrm{AS}}$$

$$\text{Now, } [\mathrm{BAX}]+[\mathrm{BAX}] = [\mathrm{ABC}] \Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4\mathrm{AS}} = s \Rightarrow p_a = \frac{4s}{2s+a}\mathrm{AS}$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}-\frac{a(b-c)^2}{4}$$

$$(a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now}, m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18}\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$\overset{?}{\geq}0\to$ true (strict inequality) $\therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_an_a\geq p_a^2\to$ ⍟

Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$

$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$

$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$

$\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2$

$\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4(R^2-2Rr+r^2)s^2$

$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0$

$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to$ true $\therefore an_a\leq 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$

$\Rightarrow\frac{n_a}{h_a}\leq\frac{R}{r}-1\to$ ⍟⍟

Now, $(2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{\text{via ⍟}}{\geq}$

$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$

$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$

$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$

$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq0\therefore(2m_a+n_a)^2\geq9p_a^2$

$\Rightarrow 2m_a+n_a\geq3p_a\therefore\frac{p_a}{h_a}+\frac{p_a}{w_a}+\frac{p_a}{m_a}\leq\frac{1}{3}\left(\frac{2m_a}{h_a}+\frac{n_a}{h_a}+\frac{2m_a}{w_a}+\frac{n_a}{w_a}+\frac{2m_a}{m_a}+\frac{n_a}{m_a}\right)$

$\leq\frac{1}{3}\left(\frac{2m_a}{h_a}+\frac{n_a}{h_a}+\frac{2m_a}{h_a}+\frac{n_a}{h_a}+\frac{2m_a}{m_a}+\frac{n_a}{h_a}\right)\overset{\text{via ⍟⍟ and Panaitopol}}{\leq}$

$\frac{1}{3}\left(\frac{R}{r}+\frac{R}{r}-1+\frac{R}{r}+\frac{R}{r}-1+2+\frac{R}{r}-1\right)=\frac{5R}{3r}-\frac{1}{3}$

$\therefore\frac{1}{h_a}+\frac{1}{w_a}+\frac{1}{m_a}\leq\left(\frac{5R}{3r}-\frac{1}{3}\right)\frac{1}{p_a}\ \forall\Delta ABC$ (QED)