ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{w_a m_a p_a}{h_a g_a n_a}+\frac{w_b m_b p_b}{h_b g_b n_b}+\frac{w_c m_c p_c}{h_c g_c n_c}\le\frac{1}{3}+\frac{4R}{3r}$$

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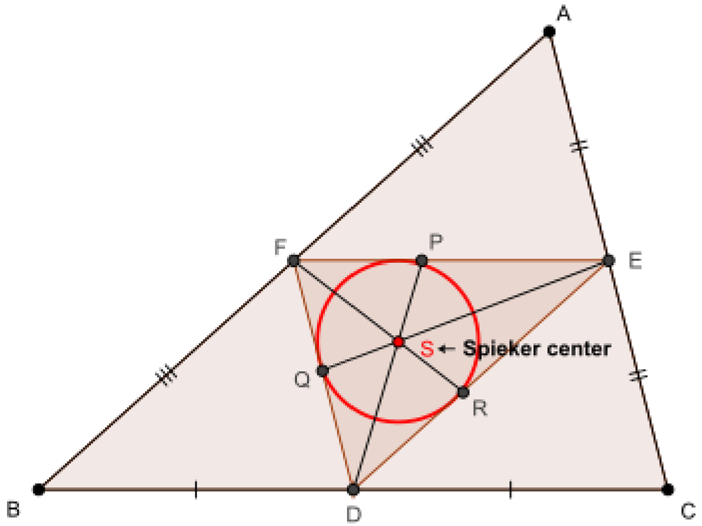

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow[DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\boxed{\overset{(**)}{=}}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\boxed{\overset{(ii)}{=}}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$
$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$
$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4} =$$
$$(2s+a).\frac{\left(\begin{array}{c}4(z+x)^2+4(x+y)^2-4(z+x)(x+y)\\ +(y+z)\left((z+x)+(x+y)-2(y+z)\right)\end{array}\right)}{4}-\frac{a(b-c)^2}{4}$$
$$(a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore b^3+c^3-abc+a\left(4m_a^2\right) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$\Rightarrow ap_a^2 \overset{(\bullet\bullet\bullet)}{=} as(s-a)-\frac{a(b-c)^2}{4}+\frac{a(4s+a)^2}{(4s+2a)^2}.(b-c)^2$$
$$\text{Now, } \frac{a(4s+a)^2}{(4s+2a)^2} = a.\frac{(4s+2a)^2-2a(4s+2a)+a^2}{(4s+2a)^2}$$
$$= a-\frac{(a+2s-2s)^2}{2s+a}+\frac{(a+2s-2s)^3}{(4s+2a)^2}$$
$$= a-(2s+a)+4s-\frac{4s^2}{2s+a}+\frac{1}{4}\left(\frac{(2s+a)^3-8s^3-3(2s+a)(2s)a}{(2s+a)^2}\right)$$
$$= 2s-\frac{4s^2}{2s+a}+\frac{2s+a}{4}-\frac{2s^3}{(2s+a)^2}-\frac{3s(a+2s-2s)}{2(2s+a)}$$
$$= \frac{5s}{2}+\frac{a}{4}-\frac{4s^2}{2s+a}-\frac{2s^3}{(2s+a)^2}-\frac{3s}{2}+\frac{3s^2}{2s+a}$$

$$\therefore \frac{a(4s+a)^2}{(4s+2a)^2} \overset{(\bullet\bullet\bullet\bullet)}{=} s+\frac{a}{4}-\frac{s^2(4s+a)}{(2s+a)^2}$$

$$\therefore (\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet) \Rightarrow ap_a^2 = as(s-a)-\frac{a(b-c)^2}{4}+s(b-c)^2+\frac{a(b-c)^2}{4}$$

$$-\frac{s^2(4s+a)}{(2s+a)^2}.(b-c)^2 \overset{a<s}{\le} as(s-a)+s(b-c)^2-\frac{s^2(4s+a)}{(2s+s)^2}.(b-c)^2$$

$$= as(s-a)+s(b-c)^2-\frac{(4s+a)(b-c)^2}{9}$$

$$\Rightarrow ap_a^2 \le as(s-a)+\frac{5s(b-c)^2}{9}-\frac{a(b-c)^2}{9} \text{ and analogs}$$

$$\therefore \sum_{cyc} ap_a^2 \le s\left(2s^2-2\left(s^2-4Rr-r^2\right)\right)+\frac{10s}{9}\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$-\frac{1}{9}\left(2s\left(s^2+4Rr+r^2\right)-9abc\right)$$

$$= s\left(8Rr+2r^2\right)+\frac{10s\left(s^2-12Rr-3r^2\right)}{9}-\frac{2s\left(s^2-14Rr+r^2\right)}{9}$$

$$= \frac{2s\left(4s^2-10Rr-7r^2\right)}{9} \Rightarrow \frac{1}{2rs}.\sum_{cyc} ap_a^2 \overset{\circledast}{\le} \frac{4s^2-10Rr-7r^2}{9r} \text{ and}$$

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \overset{CBS}{\le} \sqrt{\sum_{cyc}\frac{p_a^2}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}} = \sqrt{\frac{1}{2rs}.\sum_{cyc} ap_a^2}.\sqrt{\frac{1}{r}} \overset{\text{via }\circledast}{\le} \sqrt{\frac{4s^2-10Rr-7r^2}{9r^2}}$$

$$\overset{Gerretsen}{\le} \sqrt{\frac{4(4R^2+4Rr+3r^2)-10Rr-7r^2}{9r^2}} = \sqrt{\frac{16R^2+6Rr+5r^2}{9r^2}} \overset{Euler}{\le}$$

$$\sqrt{\frac{16R^2+6Rr+r^2+2Rr}{9r^2}} = \sqrt{\frac{(4R+r)^2}{9r^2}} \therefore \frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \le \frac{1}{3}+\frac{4R}{3r} \to (m)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$ and $b^2(s-b)+c^2(s-c) = ag_a^2+a(s-b)(s-c)$ and via summation, we get :

$$(b^2+c^2)(2s-b-c) = an_a^2+ag_a^2+2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2) =$$

$$2a\left(n_a^2+g_a^2\right)+a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) = 2\left(n_a^2+g_a^2\right)+$$

$$a^2-(b-c)^2 \Rightarrow 2\left(b^2+c^2\right)-a^2+(b-c)^2 = 2\left(n_a^2+g_a^2\right) \Rightarrow 4m_a^2+(b-c)^2$$

$$= 2\left(n_a^2+g_a^2\right) \Rightarrow 2(b-c)^2+4s(s-a) = 2\left(n_a^2+g_a^2\right)$$

$$\Rightarrow n_a^2+g_a^2 \overset{\circledcirc}{=} (b-c)^2+2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$

$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a) = an_a^2+a\left(s^2-s(2s-a)+bc\right) \Rightarrow s\left(b^2+c^2\right)-2sbc$$

$$= an_a^2+a\left(as-s^2\right) \Rightarrow s\left(b^2+c^2-a^2-2bc\right) = an_a^2-as^2 \Rightarrow an_a^2 = as^2+$$

$$s(2bc\cos A-2bc) = as^2-4sbc\sin^2\frac{A}{2} = as^2-\frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2-as\left(\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s-\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{\circledcirc\circledcirc}{=} s(s-a)+\frac{s}{a}(b-c)^2$$

Via $(\bullet)$ and $(\bullet\bullet)$, we get: $g_a^2 = (b-c)^2+2s(s-a)-s^2+\dfrac{4s(s-b)(s-c)}{a}$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)\left(s - a + \frac{a^2 - (b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{\circledcirc\circledcirc\circledcirc}{=} (s-a)\left(s - \frac{(b-c)^2}{a}\right)$$

$$\therefore (\bullet\bullet), (\bullet\bullet\bullet) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s - a + \frac{(b-c)^2}{a}\right)\left(s - \frac{(b-c)^2}{a}\right)$$

$$= s(s-a)\left(s(s-a) + s\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$

$$\Rightarrow n_a^2 g_a^2 \overset{(\blacksquare)}{=} s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

$$\text{Again, } m_a^2 w_a^2 = \frac{(b-c)^2 + 4s(s-a)}{4} \cdot \frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow m_a^2 w_a^2 \overset{(\blacksquare\blacksquare)}{=} s(s-a)\frac{bc}{(b+c)^2}\left((b-c)^2 + 4s(s-a)\right)$$

$$\therefore (\blacksquare), (\blacksquare\blacksquare) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2$$

$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2} - \frac{bc}{(b+c)^2}\left((b-c)^2 + 4s(s-a)\right)\right)$$

$$= s(s-a)\begin{pmatrix} s(s-a) + (b-c)^2\left(\frac{a^2 - (b-c)^2}{a^2}\right) \\ -\frac{bc}{(b+c)^2}\left((b-c)^2 + (b+c)^2 - a^2\right) \end{pmatrix}$$

$$= s(s-a)\left(s(s-a) - bc + \left(a^2 - (b-c)^2\right)\left(\frac{(b-c)^2}{a^2} + \frac{bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(\left((b+c)^2 - a^2 - 4bc\right) + \left(a^2 - (b-c)^2\right)\left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left((b-c)^2 - a^2 + \left(a^2 - (b-c)^2\right)\left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2}\right)\right)$$

$$= \frac{s(s-a)}{4}\left(a^2 - (b-c)^2\right)\left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} - 1\right)$$

$$= \frac{s(s-a)}{4} . 4(s-b)(s-c)\left(\frac{4(b-c)^2}{a^2} - \frac{(b-c)^2}{(b+c)^2}\right) = r^2 s^2 (b-c)^2\left(\frac{4}{a^2} - \frac{1}{(b+c)^2}\right)$$

$$= r^2 s^2 (b-c)^2\left(\frac{2}{a} + \frac{1}{b+c}\right)\left(\frac{2b + 2c - a}{a(b+c)}\right) \geq 0$$

$$\Rightarrow n_a^2 g_a^2 \geq m_a^2 w_a^2 \Rightarrow n_a g_a \geq m_a w_a \to (n)$$

$$\therefore \frac{w_a m_a p_a}{h_a g_a n_a} + \frac{w_b m_b p_b}{h_b g_b n_b} + \frac{w_c m_c p_c}{h_c g_c n_c} \overset{\text{via }(n)}{\leq} \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \overset{\text{via }(m)}{\leq} \frac{1}{3} + \frac{4R}{3r}$$

$$\therefore \frac{w_a m_a p_a}{h_a g_a n_a} + \frac{w_b m_b p_b}{h_b g_b n_b} + \frac{w_c m_c p_c}{h_c g_c n_c} \leq \frac{1}{3} + \frac{4R}{3r}\ \forall \Delta\, ABC,$$

$''='' $ iff Δ ABC is equilateral (QED)