ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$n_a \le h_a + 3(R-2r)$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2-(b-c)^2) = as(s-a) + s(b-c)^2$$
$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (1)$$
$$\text{Now}, n_a \le h_a + 3(R-2r) \overset{\text{via }(1)}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2$$
$$\le s(s-a) - \frac{s(s-a)(b-c)^2}{a^2} + 9(R-2r)^2 + \frac{2bc}{2R}.3(R-2r)$$
$$\Leftrightarrow \frac{s^2}{a^2}(b-c)^2 \le 9(R-2r)^2 + 3bc\left(\frac{R-2r}{R}\right)$$
$$\Leftrightarrow \frac{16R^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2} \le 9R^2(1-4SC+4S^2)^2 +$$
$$6R^2\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)(1-4SC+4S^2) \;\left(C = \cos\frac{B-C}{2}, S = \sin\frac{A}{2}\right)$$
$$\Leftrightarrow 4(S+C)^2.(1-C^2) \overset{(*)}{\le} 9(1-4SC+4S^2)^2 + 12(C^2-S^2)(1-4SC+4S^2) \text{ and}$$
$$\because P = 4(C-2S)^4 + 8S(1-C)(C-2S)^2 \overset{1\ge C}{\ge} 0 \therefore \text{in order to prove } (*),$$
$$\text{it suffices to prove : RHS of } (*) - \text{LHS of } (*) \overset{?}{\ge} P$$
$$\Leftrightarrow (68S^2+8-8S)C^2 - (80S^3+80S-32S^2)C + 32S^4 - 32S^3 + 56S^2 + 9 \underset{(**)}{\overset{?}{\ge}} 0$$

Now, LHS of $(**)$, a quadratic polynomial in C, has a discriminant

$$\delta = (80S^3+80S-32S^2)^2 - 4(68S^2+8-8S)(32S^4-32S^3+56S^2+9)$$
$$= -144\left(S^2+2(1-S)\right)(1-4S^2)^2 \overset{1>S}{\le} 0 \Rightarrow \text{LHS of } (**) \ge 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore n_a \le h_a + 3(R-2r)\ \forall\, \Delta\, ABC, \text{''}=\text{'' iff } \Delta\, ABC \text{ is equilateral (QED)}$$