ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $r_a = 3R + r$, the following relationship holds :

$$h_a + w_a + m_a \geq \left(3 + 2\sqrt{3}\right)r$$

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$$r_a = 3R + r \Rightarrow r_a + r_b + r_c = 3R + r + r_b + r_c$$

$$\Rightarrow 4R + r = 3R + r + 4R\cos^2\frac{A}{2} \Rightarrow \cos^2\frac{A}{2} = \frac{1}{4} \Rightarrow \frac{A}{2} = \frac{\pi}{3} \Rightarrow A = 120^\circ$$

$$\Rightarrow \frac{B}{2} + \frac{C}{2} = 30^\circ \therefore 0^\circ < \frac{B}{2} < 30^\circ \text{ and } -30^\circ < -\frac{C}{2} < 0^\circ \Rightarrow -30^\circ < \frac{B-C}{2} < 30^\circ$$

$$\Rightarrow c = \cos\frac{B-C}{2} > \frac{\sqrt{3}}{2} \to (1) \text{ and } \frac{A}{2} = \frac{\pi}{3} \Rightarrow s = \sin\frac{A}{2} = \frac{\sqrt{3}}{2} \to (2)$$

$$\text{Now, } h_a + w_a + m_a = 2R(c^2 - s^2) + \frac{4R^2\cos\frac{A}{2}.\left(2\cos^2\frac{B-C}{2} - 2\sin^2\frac{A}{2}\right)}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$+\sqrt{4R^2\cos^2\frac{A}{2}.\left(\cos^2\frac{B-C}{2} - \sin^2\frac{A}{2}\right) + \frac{16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}}{4}}$$

$$= 2R(c^2 - s^2) + \frac{2R(c^2 - s^2)}{c}$$

$$+\sqrt{4R^2\cos^2\frac{A}{2}.\left(\cos^2\frac{B-C}{2} - \sin^2\frac{A}{2}\right) + 4R^2\sin^2\frac{A}{2}\left(1 - \cos^2\frac{B-C}{2}\right)}$$

$$\overset{\text{via }(2)}{=} 2R\left(c^2 - \frac{3}{4}\right) + \frac{2R\left(c^2 - \frac{3}{4}\right)}{c} + \sqrt{R^2\left(c^2 - \frac{3}{4}\right) + 3R^2(1 - c^2)}$$

$$= r.\frac{R}{r}.\left(2\left(c^2 - \frac{3}{4}\right) + \frac{2\left(c^2 - \frac{3}{4}\right)}{c} + \frac{\sqrt{9 - 8c^2}}{2}\right)$$

$$= r.\frac{R}{2Rs(c - s)}.\left(2\left(c^2 - \frac{3}{4}\right) + \frac{2\left(c^2 - \frac{3}{4}\right)}{c} + \frac{\sqrt{9 - 8c^2}}{2}\right)$$

$$\overset{\text{via }(2)}{=} \frac{r}{\sqrt{3}}.\left(\frac{2\left(c^2 - \frac{3}{4}\right)}{c - \frac{\sqrt{3}}{2}} + \frac{2\left(c^2 - \frac{3}{4}\right)}{c\left(c - \frac{\sqrt{3}}{2}\right)} + \frac{\sqrt{9 - 8c^2}}{2\left(c - \frac{\sqrt{3}}{2}\right)}\right)$$

$$= \frac{r}{\sqrt{3}}.\left(2\left(c + \frac{\sqrt{3}}{2}\right) + \frac{2\left(c + \frac{\sqrt{3}}{2}\right)}{c} + \frac{\sqrt{9 - 8c^2}}{2c - \sqrt{3}}\right)$$

$$= \frac{r}{\sqrt{3}}.\left(\frac{\left(2c + \sqrt{3}\right)(c + 1)}{c} + \frac{\sqrt{9 - 8c^2}}{2c - \sqrt{3}}\right) = \frac{r}{\sqrt{3}}.\frac{(c + 1)(4c^2 - 3) + c.\sqrt{9 - 8c^2}}{c\left(2c - \sqrt{3}\right)}$$

$$\overset{?}{\geq} \left(3 + 2\sqrt{3}\right)r = \sqrt{3}r\left(2 + \sqrt{3}\right)$$

$$\Leftrightarrow (c+1)(4c^2-3)+c.\sqrt{9-8c^2} \overset{?}{\geq} 3c(2+\sqrt{3})(2c-\sqrt{3})$$
$$= 3c\left(4c-3+2\sqrt{3}(c-1)\right)$$
$$\Leftrightarrow c.\sqrt{9-8c^2}+6\sqrt{3}c(1-c) \overset{?}{\geq} 3c(4c-3)-(c+1)(4c^2-3)$$
$$\Leftrightarrow c.\sqrt{9-8c^2}+6\sqrt{3}c(1-c) \underset{(*)}{\overset{?}{\geq}} -4c^3+8c^2-6c+3$$

$$\text{Now}, -4c^3+8c^2-6c+3 = (4c^2+2)(1-c)+(2c-1)^2 \overset{\frac{\sqrt{3}}{2}<c\leq 1}{>} 0$$
$$\textit{and also}, 6\sqrt{3}c(1-c) \overset{\frac{\sqrt{3}}{2}<c\leq 1}{\geq} 0 \therefore \text{in order to prove } (*),$$
$$\text{it suffices to prove}: c^2(9-8c^2) \overset{?}{\geq} (-4c^3+8c^2-6c+3)^2$$
$$\Leftrightarrow -16c^6+64c^5-120c^4+120c^3-75c^2+36c-9 \overset{?}{\geq} 0$$
$$\Leftrightarrow (1-c)\left(16c^5-48c^4+72c^3-48c^2+27c-9\right) \overset{?}{\geq} 0 \textit{ and } \because 1 \geq c$$
$$\therefore \text{in order to prove it, it suffices to prove}:$$
$$16c^5-48c^4+72c^3-48c^2+27c-9 \underset{(**)}{\overset{?}{>}} 0$$

$$\text{We have, via } (1), \frac{\sqrt{3}}{2} < c \leq 1 \therefore 16c^5-48c^4+72c^3-48c^2+27c-9$$
$$\overset{c>\frac{\sqrt{3}}{2}>\frac{4}{5}}{>} 16c^5-48c^4+72c^3-48c^2+27.\frac{4}{5}-9$$
$$= \frac{40c^2(2c^3-6c^2+9c-6)+63}{5} \overset{1\geq c^2}{\geq} \frac{40c^2(2c^3-6c^2+9c-6)+63c^2}{5}$$
$$= \frac{c^2(80c^3-240c^2+360c-177)}{5}$$
$$= \frac{c^2\left(160c(c-1)^2+20(1-c)(4c^2-3)+140\left(c-\frac{117}{140}\right)\right)}{5} > 0$$
$$\because 1 \geq c > \frac{\sqrt{3}}{2} > \frac{117}{140} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore h_a+w_a+m_a \geq \left(3+2\sqrt{3}\right)r \ \forall \ \Delta \, ABC \text{ with } r_a = 3R+r,$$
$$'' ='' \text{ iff } b = c = 30^\circ \text{ (QED)}$$