ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$\min\left\{\sum_{cyc} n_a^2, \sum_{cyc} r_a^2\right\} \ge \sum_{cyc}(2m_a^2 - g_a^2) \ge \sum_{cyc}(2m_a^2 - w_a^2) \ge \sum_{cyc} p_a^2 \ge \sum_{cyc}\left(2m_a^2 - \frac{1}{2}(r_a^2 + h_a^2)\right)$$

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$$\sum_{cyc}(r_a^2 - 2m_a^2 + g_a^2) = (4R+r)^2 - 5s^2 + 3(4Rr + r^2) + \sum_{cyc}(s-a)^2 + \frac{r}{R}\sum_{cyc} ab$$

$$\overset{?}{\ge} 0 \Leftrightarrow (4R - r)s^2 \overset{?}{\le} 16R^3 + 12R^2 r + 6Rr^2 + r^3 \to \text{true} \because L - R \overset{\text{Blundon-Gerretsen}}{\le}$$

$$\frac{R(4R-r)(4R+r)^2}{4R-2r} - (16R^3 + 12R^2 r + 6Rr^2 + r^3) = \frac{-r^2(R-2r)(4R+r)}{4R-2r} \overset{\text{Euler}}{\le} 0$$

$$\therefore \sum_{cyc} r_a^2 \ge \sum_{cyc}(2m_a^2 - g_a^2) \text{ and } n_a^2 + g_a^2 \ge \frac{1}{2}(n_a + g_a)^2 \overset{\text{Fustei}}{\ge} 2m_a^2$$

$$\therefore \sum_{cyc} n_a^2 \ge \sum_{cyc}(2m_a^2 - g_a^2) \text{ and } g_a \le w_a \text{ and analogs}$$

$$\therefore \sum_{cyc}(2m_a^2 - g_a^2) \ge \sum_{cyc}(2m_a^2 - w_a^2) \;\&\; 2m_a^2 - w_a^2 - p_a^2 \overset{\text{Fustei,Ajiba}}{=}$$

$$= (b-c)^2\left(\frac{1}{2} + \frac{s(s-a)}{(2s-a)^2} - \frac{s(3s+a)}{(2s+a)^2}\right) = (b-c)^2\frac{(s-a)(16s^2 + 4sa) + a^3}{2(4s^2 - a^2)^2} \ge 0$$

$$\therefore \sum_{cyc}(2m_a^2 - w_a^2) \ge \sum_{cyc} p_a^2 \text{ and } \sum_{cyc} p_a^2 - \sum_{cyc}\left(2m_a^2 - \frac{1}{2}(r_a^2 + h_a^2)\right)$$

$$= \frac{153s^6 - (600Rr + 353r^2)s^4 - r^2(752R^2 + 448Rr + 61r^2)s^2 - r^3(256R^3 + 176R^2 r + 40Rr^2 + 3r^3)}{(9s^2 + 6Rr + r^2)^2}$$

$$-3(s^2 - 4Rr - r^2) + \frac{(4R+r)^2 - 2s^2}{2} + \frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2}{8R^2} \overset{?}{\ge} 0 \Leftrightarrow M$$

$$= 81s^8 - (1368R^2 + 540Rr - 180r^2)s^6$$
$$+(5184R^4 + 2112R^3 r - 664R^2 r^2 + 732Rr^3 + 118r^4)s^4$$
$$+r(6912R^5 + 7808R^4 r + 2800R^3 r^2 + 1112R^2 r^3 + 268Rr^4 + 20r^5)s^2 +$$
$$r^2(2304R^6 + 3328R^5 r + 1776R^4 r^2 + 624R^3 r^3 + 152R^2 r^4 + 20Rr^5 + r^6) \overset{?}{\ge} 0 \text{ and}$$

to prove $M \overset{?}{\ge} 0$, it suffices to prove :

$$M \overset{?}{\ge} \left(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3\right)^2 \Leftrightarrow -(180R^2 - 675Rr + 36r^2)s^6$$
$$+(972R^4 - 5304R^3 r - 9886R^2 r^2 + 1317Rr^3 - 92r^4)s^4$$
$$+r(12096R^5 + 61568R^4 r + 36340R^3 r^2 + 6272R^2 r^3 - 95Rr^4 - 76r^5)s^2 -$$
$$r^2\left(82368R^6 + 123584R^5 r + 77316R^4 r^2 + 25764R^3 r^3 + 4822R^2 r^4 + 481Rr^5 + 20r^6\right) \overset{?}{\underset{(*)}{\ge}} 0$$

Case 1a $T_1 = 180R^2 - 675Rr + 36r^2 \ge 0$ and $T_2 = 252R^4 - 6204R^3 r$

$+3830R^2r^2 - 753Rr^3 - 20r^4 \geq 0$ *and then* : *via* Gerretsen *and* Double – Rouche,

$$P = -T_1s^2\theta + T_2(s^2 - 16Rr + 5r^2)^2 +$$

$$4r\left(\sigma = r^5\begin{pmatrix}7920t^5 - 43510t^4 + 48251t^3 - 15579t^2\\ +1638t + 40\end{pmatrix}\right)\begin{pmatrix}s^2\\ -16Rr + 5r^2\end{pmatrix} \geq 0$$

$$\begin{pmatrix}\theta = s^4 - s^2(4R^2 + 20Rr - 2r^2)\\ +r(4R + r)^3 \leq 0 \text{ and } \sigma > 0 \ \forall\ T_2 \geq 0\end{pmatrix}\left(t = \frac{R}{r}\right) \therefore \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (t - 2)\begin{pmatrix}90000t^5 - 179520t^4 + 116342t^3\\ -24109t^2 + 1902t + 40\end{pmatrix} \overset{?}{\geq} 0 \to \text{true} \because t \geq 2$$

Case 1b $T_1 \geq 0$ & $T_2 < 0$ *and then* : $Q = -T_1s^2\theta + T_2s^2(s^2 - 4R^2 - 4Rr - 3r^2) +$

$$r^6\begin{pmatrix}1008t^6 - 192t^5 + 18268t^4 + 2100t^3\\ +8478t^2 - 2677t - 100\end{pmatrix}(s^2 - 16Rr + 5r^2) \geq 0$$

$$\therefore \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} Q \Leftrightarrow$$

$$(t - 2)\left((t - 3)\begin{pmatrix}4032t^5 - 2460t^4 + 5924t^3\\ +10616t^2 + 42382t + 125703\end{pmatrix} + 377049\right) \overset{?}{\geq} \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \text{ and } \because T_1 \geq 0 \Rightarrow t > 3 \Rightarrow (*) \text{ is true}$$

Case 2 $T_1 < 0$ *and then* : $R = -T_1(s^2 - 4R^2 - 4Rr - 3r^2)\theta$

$$-2(234R^4 + 2112R^3r - 2923R^2r^2 - 564Rr^3 + 64r^4)\theta +$$

$$16R\left(\mu = r^5\begin{pmatrix}63t^5 + 240t^4 - 5141t^3 + 5900t^2\\ -1420t + 48\end{pmatrix}\right)(s^2 - 4R^2 - 4Rr - 3r^2) \geq 0$$

$$(\because \mu < 0 \text{ whenever } T_1 < 0) \therefore \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} R$$

$$\Leftrightarrow (t - 2)\left((t - 2)\begin{pmatrix}252t^5 + 1212t^4 + 3337t^3\\ +7635t^2 + 11016t + 23980\end{pmatrix} + 47872\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\Rightarrow (*) \text{ is true } \forall\ \Delta\ ABC \therefore \sum_{cyc} p_a^2 \geq \sum_{cyc}\left(2m_a^2 - \frac{1}{2}(r_a^2 + h_a^2)\right) \text{ and so,}$$

$$\min\left\{\sum_{cyc} n_a^2, \sum_{cyc} r_a^2\right\} \geq \sum_{cyc}(2m_a^2 - g_a^2) \geq \sum_{cyc}(2m_a^2 - w_a^2) \geq \sum_{cyc} p_a^2$$

$$\geq \sum_{cyc}\left(2m_a^2 - \frac{1}{2}(r_a^2 + h_a^2)\right) \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$