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In ΔABC the following relationship holds:

$$\frac{15}{2}+\sum\frac{\sin^2 A}{\sin^2 B+\sin^2 C}\le 9\left(\frac{R}{2r}\right)^2$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$6=\frac{6R^2}{R^2}\overset{Euler}{\le}6\left(\frac{R}{2r}\right)^2\ (1)$$

$$\sum\frac{\sin^2 A}{\sin^2 B+\sin^2 C}=\sum\frac{a^2}{b^2+c^2}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)=\frac{1}{4}\sum\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)=$$

$$=\frac{1}{4}\sum\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right)\overset{Bandila}{\le}\frac{1}{4}\sum\left(\left(\frac{R}{r}\right)^2-2\right)=\frac{3}{4}\left(\frac{R}{r}\right)^2-\frac{6}{4}=3\left(\frac{R}{2r}\right)^2-\frac{3}{2}\ (2)$$

$$\frac{15}{2}+\sum\frac{\sin^2 A}{\sin^2 B+\sin^2 C}\overset{(2)}{\le}\frac{15}{2}+3\left(\frac{R}{2r}\right)^2-\frac{3}{2}=3\left(\frac{R}{2r}\right)^2+6\overset{(1)}{\le}$$

$$\le 3\left(\frac{R}{2r}\right)^2+6\left(\frac{R}{2r}\right)^2=9\left(\frac{R}{2r}\right)^2$$

$$\textit{Equality holds for } A=B=C$$