ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}+\frac{1}{b^2}\sqrt{\frac{c^2+a^2}{c^2+ca+a^2}}+\frac{1}{c^2}\sqrt{\frac{a^2+b^2}{a^2+ab+b^2}}\le\frac{1}{4r^2}\cdot\sqrt{\frac{R}{3r}}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}=\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+c^2+bc}}\overset{AM-GM}{\le}\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{3bc}}=$$

$$=\frac{1}{a^2}\sqrt{\frac{1}{3}\left(\frac{b}{c}+\frac{c}{b}\right)}\overset{Bandila}{\le}\frac{1}{a^2}\sqrt{\frac{R}{3r}}\ (1)$$

$$\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}+\frac{1}{b^2}\sqrt{\frac{c^2+a^2}{c^2+ca+a^2}}+\frac{1}{c^2}\sqrt{\frac{a^2+b^2}{a^2+ab+b^2}}=$$

$$=\sum\frac{1}{a^2}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}\overset{(1)}{\le}\sum\frac{1}{a^2}\sqrt{\frac{R}{3r}}\overset{Steining}{\le}\sqrt{\frac{R}{3r}}\cdot\frac{1}{4r^2}$$

Equality holds for an equilateral triangle.