ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{2r+h_a} \le \frac{3}{5r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x = \frac{r}{h_a}, y = \frac{r}{h_b}, z = \frac{r}{h_c}\ then\ x+y+z = r\sum \frac{1}{h_a} = \frac{r}{r} = 1$$

$$We\ need\ to\ show:$$

$$\sum \frac{1}{2r+h_a} \le \frac{3}{5r}\ or \sum \frac{r}{2r+h_a} \le \frac{3}{5}\ or \sum \frac{\frac{r}{h_a}}{2\frac{r}{h_a}+1} \le \frac{3}{5}\ or\ \sum \frac{x}{2x+1} \le \frac{3}{5}$$

Lemma :

$$\frac{x}{2x+1} \le \frac{2+9x}{25}\ \forall\, x > 0$$

Proof:

$$\frac{x}{2x+1} \le \frac{2+9x}{25}\ \ or\ 25x \le (2x+1)(2+9x)\ or\ 18x^2 - 12x + 2 \ge 0\ or$$
$$2(3x-1)^2 \ge 0\ (true)$$

$$\sum \frac{x}{2x+1} \overset{Lemma}{\le} \sum \frac{2+9x}{25} = \frac{1}{25}\big(3\times 2 + 9(x+y+z)\big) \overset{x+y+z=1}{=} \frac{1}{25}(6+9) = \frac{15}{25} = \frac{3}{5}$$

Equality holds for an equilateral triangle.