ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$6 \leq \sum \frac{h_a + r}{h_a - r} \leq \frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{h_a + r}{h_a - r} = \frac{r\left(\frac{2s}{a} + 1\right)}{r\left(\frac{2s}{a} - 1\right)} = \frac{2s + a}{2s - a} = \frac{2a + (b + c)}{b + c} = \frac{2a}{b + c} + 1$$

$$\sum \frac{h_a + r}{h_a - r} = \sum \left(\frac{2a}{b + c} + 1\right) = 3 + 2\sum \left(\frac{a}{b + c}\right) \overset{Nesbitt}{\geq} 3 + 2.\frac{3}{2} = 6$$

$$\sum \frac{h_a + r}{h_a - r} = \sum \left(\frac{2a}{b + c} + 1\right) = 3 + 2\sum \left(\frac{a}{b + c}\right) \overset{AM-HM}{\leq} 3 + \frac{2}{4}\sum \left(\frac{a}{b} + \frac{a}{c}\right) =$$

$$= 3 + \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) \overset{Bandila}{\leq} 3 + \frac{1}{2}\sum \frac{R}{r} = 3 + \frac{3R}{2r} = \frac{3R}{R} + \frac{3R}{2r} \overset{Euler}{\leq} \frac{3R}{2r} + \frac{3R}{2r} = \frac{3R}{r}$$

Equality holds for an equilateral triangle.