ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{h_a + r}{h_a - r} \leq \sum \frac{r_a + r}{r_a - r}$$

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$$\frac{h_a + r}{h_a - r} = \frac{r\left(\frac{2s}{a} + 1\right)}{r\left(\frac{2s}{a} - 1\right)} = \frac{2s + a}{2s - a} = \frac{2a + (b + c)}{b + c} = \frac{2a}{b + c} + 1$$

$$⍰ \sum \frac{h_a + r}{h_a - r} = \sum \left(\frac{2a}{b + c} + 1\right) = 3 + 2\sum \left(\frac{a}{b + c}\right) \overset{Nesbitt}{\geq} 3 + 2 \cdot \frac{3}{2} = 6$$

$$\sum \frac{h_a + r}{h_a - r} = \sum \left(\frac{2a}{b + c} + 1\right) = 3 + 2\sum \left(\frac{a}{b + c}\right) \overset{AM-HM}{\leq}$$

$$\leq 3 + \frac{2}{4}\sum \left(\frac{a}{b} + \frac{a}{c}\right) = 3 + \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) \ (1)$$

$$\sum \frac{r_a + r}{r_a - r} = \sum \frac{\frac{rs}{s-a} + r}{\frac{rs}{s-a} - r} = \sum \frac{2s - a}{a} = \sum \frac{b + c}{a} = \sum \left(\frac{b}{a} + \frac{c}{a}\right) = \sum \left(\frac{a}{b} + \frac{b}{a}\right) =$$

$$= \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) + \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) \overset{AM-GM}{\geq} \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) + \frac{1}{2}(2 + 2 + 2) =$$

$$= 3 + \frac{1}{2}\sum \left(\frac{a}{b} + \frac{b}{a}\right) \ (2)$$

$$From\ (1) \& \ (2) we\ get \sum \frac{h_a + r}{h_a - r} \leq \sum \frac{r_a + r}{r_a - r}$$

Equality holds for an equilateral triangle.