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In ΔABC the following relationship holds:

$$h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)\geq 12\sqrt{3}r$$

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Solution by Daniel Sitaru-Romania

$$a\leq b\leq c\to\frac{1}{a}\geq\frac{1}{b}\geq\frac{1}{c}\to\frac{2F}{a}\geq\frac{2F}{b}\geq\frac{2F}{c}\to h_a\geq h_b\geq h_c$$

$$a\leq b\leq c\to\cos\frac{A}{2}\geq\cos\frac{B}{2}\geq\cos\frac{C}{2}\to\frac{1}{\cos\frac{A}{2}}\leq\frac{1}{\cos\frac{B}{2}}\leq\frac{1}{\cos\frac{C}{2}}\to$$

$$\to\sec\frac{A}{2}\leq\sec\frac{B}{2}\leq\sec\frac{C}{2}\to\begin{cases}\sec\frac{A}{2}+\sec\frac{B}{2}\leq\sec\frac{B}{2}+\sec\frac{C}{2}\\ \sec\frac{A}{2}+\sec\frac{C}{2}\leq\sec\frac{B}{2}+\sec\frac{C}{2}\\ \sec\frac{A}{2}+\sec\frac{B}{2}\leq\sec\frac{A}{2}+\sec\frac{C}{2}\end{cases}\to$$

$$\sec\frac{B}{2}+\sec\frac{C}{2}\geq\sec\frac{A}{2}+\sec\frac{C}{2}\geq\sec\frac{A}{2}+\sec\frac{B}{2}$$

$$\sum_{cyc}h_a\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)\overset{CEBYSHEV}{\geq}\frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)=$$

$$=\frac{1}{3}\sum_{cyc}\frac{2F}{a}\cdot 2\sum_{cyc}\sec\frac{A}{2}\overset{JENSEN}{\geq}\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\cdot 3\sec\left(\frac{A+B+C}{6}\right)=$$

$$=\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\cdot 3\sec\frac{\pi}{6}\overset{GERRETSEN}{\geq}\frac{1}{R\cdot\cos\frac{\pi}{6}}\cdot(16Rr-5r^2+r^2+4Rr)=$$

$$=\frac{2}{\sqrt{3}R}\cdot(20Rr-4r^2)\geq 12\sqrt{3}r\Leftrightarrow 40Rr-8r^2\geq 36Rr\Leftrightarrow$$

$$\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\geq 2r\ (EULER)$$

Equality holds for $a=b=c$.