ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$r_a^2+r_b^2+r_c^2\geq m_a^2+m_b^2+m_c^2+\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

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$$r_a^2+r_b^2+r_c^2=\left(\sum r_a\right)^2-2\sum r_ar_b=2(4R+r)^2-2s^2,\sum m_a^2=\frac{3}{2}(s^2-r^2-4Rr)$$

$$We \;\; will\; show:$$

$$r_a^2+r_b^2+r_c^2\geq m_a^2+m_b^2+m_c^2+2(R^2-4r^2)\ (A)$$

$$(4R+r)^2-2s^2\geq\frac{3}{2}(s^2-r^2-4Rr)+2(R^2-4r^2)$$

$$2(4R+r)^2-4s^2\geq 3(s^2-r^2-4Rr)+4(R^2-4r^2)$$

$$2(16R^2+8Rr+r^2)-4s^2\geq 3(s^2-r^2-4Rr)+4(R^2-4r^2)$$

$$28R^2+28Rr+21r^2\geq 7s^2\ or, 7(4R^2+4Rr+3r^2)\geq 7s^2\ true\ (Gerretsen)$$

$$\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)=\frac{1}{2}(a^2+b^2+c^2-ab-bc-ca)$$

$$=\frac{1}{2}(2s^2-2r^2-8Rr-s^2-r^2-4Rr)=\frac{1}{2}(s^2-12Rr-3r^2)\overset{(Gerretsen)}{\leq}$$

$$\leq\frac{1}{2}(4R^2+4Rr+3r^2-12Rr-3r^2)=2(R^2-2Rr)\overset{Euler}{\leq}2(R^2-4r^2)\ (B)$$

$$From\ (A)\&(B)\ we\;\; get:$$

$$r_a^2+r_b^2+r_c^2\geq m_a^2+m_b^2+m_c^2+\frac{1}{4}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Equality holds for an equilateral triangle.