ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cos A+\cos B+\cos C+\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\geq\frac{15}{2}$$

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Solution by Tapas Das-India

$$\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)=\sqrt{3}\,2R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=$$

$$=\sqrt{3}\,2R\frac{s^2+r^2+4Rr}{4Rrs}\overset{Gerretsen\ \&\ Mitrinovic}{\geq}\sqrt{3}\cdot\frac{16Rr-5r^2+r^2+4Rr}{\frac{2r3\sqrt{3}R}{2}}=$$

$$=\frac{(20Rr-4r^2)}{3Rr}=\frac{20}{3}-\frac{4r}{R}$$

$$\cos A+\cos B+\cos C+\sqrt{3}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\geq 1+\frac{r}{R}+\frac{20}{3}-\frac{4r}{R}=$$

$$=\frac{23}{3}-\frac{r}{3R}\overset{Euler}{\geq}\frac{23}{3}-\frac{1}{6}=\frac{45}{6}=\frac{15}{2}$$

Equality holds for an equilateral triangle.