ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sin A+\sin B+\sin C+\frac{2}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\geq\frac{59\sqrt{3}}{18}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\,\frac{R}{s}=x\overset{Mitrinovic}{\geq}\frac{2s}{3\sqrt{3}}\cdot\frac{1}{s}=\frac{2}{3\sqrt{3}}$$

$$We\ need\ to\ show:$$

$$\sin A+\sin B+\sin C+\frac{2}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\geq\frac{59\sqrt{3}}{18}$$

$$or,\frac{s}{R}+\frac{8R}{s}\geq\frac{59\sqrt{3}}{18}$$

$$or,18(8R^2+s^2)\geq 59\sqrt{3}Rs\ or,$$

$$144R^2-59\sqrt{3}Rs+18s^2\geq 0\ or,144x^2-59\sqrt{3}x+18\overset{\frac{R}{s}=x}{\geq}0$$

$$We\ take\ f(x)=144x^2-59\sqrt{3}x+18\,,f'(x)=288x-59\sqrt{3}>0$$

$$\left(as\ \ x\geq\frac{2}{3\sqrt{3}}\ and\ 288\cdot\frac{2}{3\sqrt{3}}-59\sqrt{3}=64\sqrt{3}-59\sqrt{3}>0\right)$$

$$so\ f(x)\ is\ an\ increasing\ function$$

$$and\ f\left(\frac{2}{3\sqrt{3}}\right)=144\left(\frac{2}{3\sqrt{3}}\right)^2-59\sqrt{3}.\frac{2}{3\sqrt{3}}+18=\frac{64}{3}-\frac{118}{3}+18=0$$

$$We\ can\ say\ f(x)\geq f\left(\frac{2}{3\sqrt{3}}\right)\ or,f(x)\geq 0\ or,$$

$$144x^2-59\sqrt{3}x+18\geq 0$$

Equality holds for $A=B=C$.