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In ΔABC the following relationship holds:

$$r_a sec\frac{A}{2}+r_b sec\frac{B}{2}+r_c sec\frac{C}{2}\geq 6\sqrt{3}r$$

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$$In\ \ \Delta ABC\ \ wlog\ \ a\geq b\geq c\geq\ \to\ \frac{A}{2}\geq\frac{B}{2}\geq\frac{C}{2}\ \to cos\frac{A}{2}\leq cos\frac{B}{2}\leq cos\frac{C}{2}$$

$$\boxed{sec\frac{A}{2}\geq sec\frac{B}{2}\geq sec\frac{C}{2}}\ (1)$$

$$and\ a\geq b\geq c\ \ \boxed{r_a\geq r_b\geq r_c}\ (2)$$

$$Then, according\ \ to\ \ Chebyshev's\ \ theorem:$$

$$r_a sec\frac{A}{2}+r_b sec\frac{B}{2}+r_c sec\frac{C}{2}\geq\frac{1}{3}(r_a+r_b+r_c)\left(sec\frac{A}{2}+sec\frac{B}{2}+sec\frac{C}{2}\right)=$$

$$=\frac{1}{3}s\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)\left(sec\frac{A}{2}+sec\frac{B}{2}+sec\frac{C}{2}\right)\overset{A-G}{\geq}$$

$$\geq\frac{1}{3}s\cdot 3\left(\prod_{cyc}tan\frac{A}{2}\right)^{\frac{1}{3}}\cdot 3\left(\prod_{cyc}\frac{1}{cos\frac{A}{2}}\right)^{\frac{1}{3}}=3s\cdot\left(\prod_{cyc}\frac{1}{cos\frac{A}{2}}\cdot\prod_{cyc}tan\frac{A}{2}\right)^{\frac{1}{3}}=$$

$$=3s\cdot\left(\frac{r}{s}\cdot\frac{1}{\frac{s}{4R}}\right)^{\frac{1}{3}}=3s\left(\frac{4Rr}{s^2}\right)^{\frac{1}{3}}=3\left(\frac{4Rr.s^3}{s^2}\right)^{\frac{1}{3}}=3(4Rrs)^{\frac{1}{3}}\overset{Euler\&\ Mitrinovici}{\geq}$$

$$\geq 3\left(4\cdot 2r\cdot r.3\sqrt{3}\cdot r\right)^{\frac{1}{3}}=3\cdot 2r\cdot\left(3\sqrt{3}\right)^{\frac{1}{3}}=6\sqrt{3}r$$

$$r_a sec\frac{A}{2}+r_b sec\frac{B}{2}+r_c sec\frac{C}{2}\geq 6\sqrt{3}r\ \ (Proved)$$