ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cos A+\cos B+\cos C+2\left(\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\right)\geq\frac{19}{2}$$

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$$Let\ x=\cos A, y=\cos B, z=\cos C$$

$$\cos A+\cos B+\cos C=x+y+z=1+\frac{r}{R}\overset{Euler}{\leq}\frac{3}{2}$$

$$\cos A+\cos B+\cos C+2\left(\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\right)=\sum\left(\cos A+2\sec^2\frac{A}{2}\right)$$

$$=\sum\left(\cos A+\frac{4}{2\cos^2\frac{A}{2}}\right)=\sum\left(\cos A+\frac{4}{1+\cos A}\right)=\sum\left(x+\frac{4}{1+x}\right)$$

$$Lemma: x+\frac{4}{1+x}\geq\frac{(32-7x)}{9}\ \forall x\in\left(0,\frac{3}{2}\right)$$

Proof:

$$9x+9x^2+36\geq 32+32x-7x-7x^2$$

$$16x^2-16x+4\geq 0\ or, (4x-2)^2\geq 0\ true$$

$$\cos A+\cos B+\cos C+2\left(\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\right)=\sum\left(\cos A+2\sec^2\frac{A}{2}\right)=$$

$$=\sum\left(\cos A+\frac{4}{2\cos^2\frac{A}{2}}\right)=\sum\left(\cos A+\frac{4}{1+\cos A}\right)=\sum\left(x+\frac{4}{1+x}\right)\geq\sum\frac{32-7x}{9}=$$

$$=\frac{96-7(x+y+z)}{9}\geq\frac{96-\frac{7.3}{2}}{9}=\frac{192-21}{18}=\frac{171}{18}=\frac{19}{2}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{2}\ or\ A=B=C=\frac{\pi}{3}.$$