ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sin(A)\cos(B)+\sin(B)\cos(C)+\sin(C)\cos(A)\le\frac{3\sqrt{3}}{4}$$

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$$\sin(A)\cos(B)+\sin(B)\cos(C)+\sin(C)\cos(A)=\frac{1}{2}(\sin(A+B)+\sin(A-B))+$$

$$\frac{1}{2}(\sin(B+C)+\sin(B-C))+\frac{1}{2}(\sin(A+C)+\sin(A-C))=\frac{1}{2}(\sin(A+B)+\sin(B+C)+$$

$$\sin(A+C))+\frac{1}{2}(\sin(A-B)+\sin(B-C)+\sin(C-A))=\frac{1}{2}(sinA+sinB+sinC)+$$

$$\frac{1}{2}(\sin(A-B)+\sin(B-C)+\sin(C-A))=\frac{1}{2}\left(\sum\nolimits_1+\sum\nolimits_2\right)$$

$$*\sum\nolimits_2=\sin(A-B)+\sin(B-C)+\sin(C-A)=2sin\frac{(A-B)+(B-C)}{2}.cos\frac{(A-B)-(B-C)}{2}+$$

$$\sin(C-A)=2sin\frac{A-C}{2}.cos\frac{(A+C)-2B}{2}-\sin(A-C)=2sin\frac{A-C}{2}.cos\frac{(\pi-b)-2B}{2}-$$

$$2sin\frac{A-C}{2}.cos\frac{A-C}{2}=2sin\frac{A-C}{2}.sin\frac{3B}{2}-2sin\frac{A-C}{2}.cos\frac{A-C}{2}=2sin\frac{A-C}{2}(sin\frac{3B}{2}-$$

$$cos\frac{A-C}{2})=2sin\frac{A-C}{2}\left(sin\frac{3B}{2}-\sin\left(\frac{\pi}{2}-\frac{A-C}{2}\right)\right)=$$

$$2sin\frac{A-C}{2}.2sin\frac{\frac{3B}{2}-\frac{\pi}{2}+\frac{A-C}{2}}{2}.cos\frac{\frac{3B}{2}+\frac{\pi}{2}-\frac{A-C}{2}}{2}=$$

$$4sin\frac{A-C}{2}.sin\frac{3B-\pi+A-C}{4}.cos\frac{3B+\pi-A+C}{4}=$$

$$4sin\frac{A-C}{2}.sin\frac{2B+(A+B)-\pi-C}{4}.cos\frac{2B+(C+B)+\pi-A}{4}=$$

$$4sin\frac{A-C}{2}.sin\frac{2B+(\pi-C)-\pi-C}{4}.cos\frac{2B+(\pi-A)+\pi-A}{4}=$$

$$4sin\frac{A-C}{2}.sin\frac{2B-2C}{4}.cos\frac{2B+2\pi-2A}{4}=4sin\frac{A-C}{2}.sin\frac{B-C}{2}.\cos\left(\frac{\pi}{2}+\frac{B-A}{2}\right)=$$

$$4sin\frac{A-C}{2}.sin\frac{B-C}{2}.\left(-sin\frac{B-A}{2}\right)=-4sin\frac{A-C}{2}.sin\frac{B-C}{2}.sin\frac{B-A}{2}=$$

$$-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}$$

$$\boxed{\sum\nolimits_2 = -4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}}$$

$$*\sum\nolimits_1 = sinA + sinB + sinC \le \frac{3\sqrt{3}}{2} \quad Let's\ prove\ it.$$

$$Let\ \ f(x) = \sin(x) \quad x \in [0;180°] \quad f''(x) = \sin(x) < 0$$

According to Jensen's theorem.

$$\frac{sinA + sinB + sinC}{3} \le \sin\left(\frac{A+B+C}{3}\right) \quad sinA + sinB + sinC \le 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2} \quad (1)$$

In triangle ΔABC *wlog*: $A \le B \le C$

$$\sum\nolimits_2 = -4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2} \le 0 \quad (2)$$

So ,

$$\sin(A)\cos(B) + \sin(B)\cos(C) + \sin(C)\cos(A) = \frac{1}{2}(sinA + sinB + sinC) +$$

$$\left(-4sin\frac{A-B}{2}.sin\frac{B-C}{2}.sin\frac{C-A}{2}\right) \overset{2}{\le} \frac{1}{2}(sinA + sinB + sinC) \overset{1}{\le} \frac{3\sqrt{3}}{4} \quad (Proved)$$

Equality holds if $a = b = c$