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In acute ΔABC the following relationship holds:

$$\frac{1}{\cos A\cos B}+\frac{1}{\cos B\cos C}+\frac{1}{\cos C\cos A}\ge\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tpas Das-India

$$\sum\cos A\cos B=\frac{s^2+r^2-4R^2}{4R^2}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2-4R^2}{4R^2}=\frac{Rr+r^2}{R^2}\quad(1)$$

$$\frac{1}{\cos A\cos B}+\frac{1}{\cos B\cos C}+\frac{1}{\cos C\cos A}\overset{CBS}{\ge}\frac{(1+1+1)^2}{\sum\cos A\cos B}\overset{(1)}{\ge}\frac{9R^2}{Rr+r^2}$$

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}}=\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\sum\sin\frac{A}{2}\overset{Jensen}{\le}$$

$$\le\frac{4R}{r}.3\sin\left(\frac{A+B+C}{6}\right)=\frac{4R}{r}.\frac{3}{2}$$

$$We\ need\ to\ show:$$

$$\frac{9R^2}{Rr+r^2}\ge\frac{4R}{r}.\frac{3}{2}\ or,\frac{3R}{R+r}\ge2\ \ or,R\ge2r\quad EULER$$

Equality holds for an equilateral triangle.