In any Δ ABC, the following relationship holds :

$$\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}\geq\sqrt{3}$$

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$$\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}=\sum_{cyc}\frac{a}{\frac{2rs}{a}+\frac{rs}{s-a}}=\sum_{cyc}\frac{a^2(s-a)}{rs(b+c)}$$

$$=\sum_{cyc}\frac{a^2(2s-a-s)}{rs(b+c)}=\frac{1}{rs}.\sum_{cyc}a^2-\frac{1}{r}\left(\sum_{cyc}\frac{4s^2}{b+c}+\sum_{cyc}\frac{(2s-a)^2}{2s-a}-\sum_{cyc}\frac{4s(2s-a)}{2s-a}\right)$$

$$=\frac{2(s^2-4Rr-r^2)}{rs}-\frac{1}{r}\left(\frac{4s^2(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}+4s-12s\right)$$

$$=\frac{2(s^2-4Rr-r^2)}{rs}-\frac{2s(s^2-4Rr-3r^2)}{r(s^2+2Rr+r^2)}$$

$$=\frac{2\left((2R+3r)s^2-r(8R^2+6Rr+r^2)\right)}{s(s^2+2Rr+r^2)}\overset{?}{\geq}\sqrt{3}$$

$$\Leftrightarrow\frac{4\left((2R+3r)s^2-r(8R^2+6Rr+r^2)\right)^2}{s^2(s^2+2Rr+r^2)^2}\overset{?}{\geq}3$$

$$\Leftrightarrow -3s^6+(16R^2+36Rr+30r^2)s^4-r(128R^3+300R^2r+172Rr^2+27r^3)s^2$$
$$+r^2(256R^4+384R^3r+208R^2r^2+48Rr^3+4r^4)\overset{?}{\geq}0 \quad (1)$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=$
$2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$
$$\Rightarrow -3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0 \text{ and so,}$$

in order to prove (1), it suffices to prove :

$$\text{LHS of } (1)\overset{?}{\geq}-3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$\Leftrightarrow (R-3r)^2s^4+r(16R^3-39R^2r-34Rr^2-6r^3)s^2$$
$$+r^2(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)\overset{?}{\geq}0 \quad (2) \text{ and}$$

$\because (R-3r)^2(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore$ in order to prove (2),
it suffices to prove : LHS of (2) $\overset{?}{\geq}(R-3r)^2(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (48R^3-241R^2r+314Rr^2-96r^3)s^2\overset{?}{\geq} \quad (3)$$

$$r(192R^4-1792R^3r+3237R^2r^2-1602Rr^3+224r^4)$$

Case 1 $48R^3-241R^2r+314Rr^2-96r^3\geq 0$ *and then* : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$(48R^3-241R^2r+314Rr^2-96r^3)(16Rr-5r^2)\overset{?}{\geq}\text{RHS of ③}$$

$$\Leftrightarrow 36t^4-144t^3+187t^2-94t+16\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(36t^2(t-2)+43t-8)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow\text{③ is true}$$

Case 2 $48R^3-241R^2r+314Rr^2-96r^3<0$ *and then* : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$(48R^3-241R^2r+314Rr^2-96r^3)(4R^2+4Rr+3r^2)\overset{?}{\geq}\text{RHS of ③}$$

$$\Leftrightarrow 48t^5-241t^4+557t^3-772t^2+540t-128\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\Big((t-2)(48t^3-49t^2+169t+100)+264\Big)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true $\forall\,\Delta$ ABC

$$\therefore\frac{a}{h_a+r_a}+\frac{b}{h_b+r_b}+\frac{c}{h_c+r_c}\geq\sqrt{3}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$