ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}\le\frac{3}{4}$$

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$$\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}=$$

$$=\frac{1}{\prod_{cyc}\cos\frac{B-C}{2}}\cdot\sum_{cyc}\left(\cos B\cos C\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)\le$$

$$\le\frac{1}{\prod_{cyc}\left(\frac{b+c}{a}.\sin\frac{A}{2}\right)}\cdot\sum_{cyc}\cos B\cos C$$

$$\left(\because 0<\cos\frac{B-C}{2}\le 1\ and\ analogs\ and\ \cos B\cos C\ and\ analogs>0\right)$$

$$=\frac{4Rrs}{4s(s^2+2Rr+r^2).\frac{r}{4R}}\cdot\left(\left(\frac{R+r}{R}\right)^2-3+\frac{s^2-4Rr-r^2}{2R^2}\right)$$

$$=\frac{4R^2}{s^2+2Rr+r^2}\cdot\frac{s^2-4R^2+r^2}{2R^2}=\frac{2(s^2-4R^2+r^2)}{s^2+2Rr+r^2}\overset{?}{\le}\frac{3}{4}$$

$$\Leftrightarrow 32R^2+6Rr-5r^2\underset{①}{\underbrace{\overset{?}{\ge}}}5s^2$$

$$\text{Now, }5s^2\overset{\text{Gerretsen}}{\le}20R^2+20Rr+15r^2\overset{?}{\le}32R^2+6Rr-5r^2$$

$$\Leftrightarrow 6R^2-7Rr-10r^2\overset{?}{\ge}0\Leftrightarrow(6R+5r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\Rightarrow ①\text{ is true}\therefore\frac{\cos B\cos C}{\cos\frac{B-C}{2}}+\frac{\cos C\cos A}{\cos\frac{C-A}{2}}+\frac{\cos A\cos B}{\cos\frac{A-B}{2}}\le\frac{3}{4}$$

$$\forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$