ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{b}{a^2c^2}\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)\geq\frac{6\sin^3 B}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

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Solution by Tapas Das-India

$$\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)=\left(\sum a^2\right)\left(\sum a\right)-\sum a^3=$$

$$=2s(2s^2-2r^2-8Rr-s^2+3r^2+6Rr)=$$

$$=2s(s^2+r^2-2Rr)\overset{Gerretsen}{\geq}2s(16Rr-5r^2+r^2-2Rr)=2s(14Rr-4r^2)$$

$$\frac{6\sin^3 B}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{6b^3}{8R^3}\cdot\frac{4R}{s}=\frac{3b^3}{R^2s}$$

We need to show:

$$\frac{b}{a^2c^2}2s(14Rr-4r^2)\geq\frac{3b^3}{R^2s}\ \ or\ \ \frac{2s^2R^2}{a^2b^2c^2}(14Rr-4r^2)\geq 3$$

$$\frac{2s^2R^2}{16R^2r^2s^2}(14Rr-4r^2)\geq 3\ or\ (14Rr-4r^2)\geq 24r^2\ or$$

$$14Rr\geq 28r^2\ or\ R\geq 2r\ true.$$

Equality holds for an equilateral triangle.