ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}}+\frac{1}{\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}}+\frac{1}{\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}}\ge\frac{3}{\left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}}$$

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Solution by Tapas Das-India

$$\sum\cos\frac{A}{2}\overset{Jensen}{\le}3\cos\left(\frac{A+B+C}{6}\right)=3\cos\frac{\pi}{6}=\frac{3\sqrt{3}}{2}\ (1)$$

$$\sum\sin A=\frac{s}{R}\overset{Mitrinovic}{\le}\frac{3\sqrt{3}}{2}\ (2)$$

$$\sum\cos\frac{A}{2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\cos\frac{A}{2}}=3\sqrt[3]{\frac{s}{4R}}$$

$$\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}\cdot\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}\cdot\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}\overset{AM-GM}{\le}$$

$$\le\left(\frac{\sin\frac{A}{2}\cos\frac{A}{2}+\sin\frac{B}{2}\cos\frac{B}{2}+\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\sum\cos\frac{A}{2}}=$$

$$=\left(\frac{1}{2}\frac{\sin A+\sin B+\sin C}{\sum\cos\frac{A}{2}}\right)^{\sum\cos\frac{A}{2}}\overset{(1),(2),(3)}{\le}\left(\frac{1}{2}\frac{s}{R}\frac{1}{3}\cdot\sqrt[3]{\left(\frac{4R}{s}\right)}\right)^{\frac{3\sqrt{3}}{2}}=$$

$$=\left(\frac{1}{6}\left(\frac{s}{R}\right)^{\frac{2}{3}}2^{\frac{2}{3}}\right)^{\frac{3\sqrt{3}}{2}}\overset{(2)}{\le}\left(\frac{1}{6}\left(\frac{3\sqrt{3}}{2}\right)^{\frac{2}{3}}2^{\frac{2}{3}}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{1}{2}\right)^{\frac{3\sqrt{3}}{2}}\ (4)$$

$$\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}}+\frac{1}{\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}}+\frac{1}{\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}}\overset{AM-GM}{\ge}$$

$$\ge3\sqrt[3]{\frac{1}{\left(\sin\frac{A}{2}\right)^{\cos\frac{A}{2}}\cdot\left(\sin\frac{B}{2}\right)^{\cos\frac{B}{2}}\cdot\left(\sin\frac{C}{2}\right)^{\cos\frac{C}{2}}}}\overset{(4)}{\ge}3\left(\frac{1}{\left(\frac{1}{2}\right)^{\frac{3\sqrt{3}}{2}}}\right)^{\frac{1}{3}}=\frac{3}{\left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}}$$

Equality holds for an equilateral triangle.