ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\ge\frac{1}{2}+\frac{2r}{R}$$

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$$\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}=$$

$$=\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.\sum_{\text{cyc}}\left(bc\left(\sum_{\text{cyc}}a^2b^2+a^4\right)\right)$$

$$=\frac{\left(\sum_{\text{cyc}}a^2b^2\right)\left(\sum_{\text{cyc}}ab\right)-abc\sum_{\text{cyc}}a^3}{\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^2b^2\right)-a^2b^2c^2}$$

$$=\frac{\left((s^2+4Rr+r^2)^2-16Rrs^2\right)(s^2+4Rr+r^2)-8Rrs^2(s^2-6Rr-3r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}\overset{?}{\ge}\frac{R+4r}{2R}$$

$$\Leftrightarrow -2s^6+(8R^2+25Rr-2r^2)s^4-r(52R^3+92R^2r+14Rr^2-2r^3)s^2$$

$$+r^2(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\ \boxed{\overset{?}{\ge}\ (1)}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$,

where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\ \boxed{\overset{(*)}{\le}}\ 0$$

Via $(*)$ and Gerretsen, $P=-2s^2\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)$

$-r(15R-2r)s^2(s^2-4R^2-4Rr-3r^2)\ge 0\ \therefore$ in order to prove (1),

it suffices to prove : LHS of (1) $\overset{?}{\ge}$ P $\Leftrightarrow (16R^3-48R^2r-27Rr^2+10r^3)s^2$

$$+r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\ \boxed{\overset{?}{\ge}\ (2)}\ 0$$

Case 1 $16R^3-48R^2r-27Rr^2+10r^3\ge 0$ and then : LHS of (2) $\ge$
$r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)>0\Rightarrow$ (2) is true

Case 2 $16R^3-48R^2r-27Rr^2+10r^3<0$ and then : LHS of (2)

$$\overset{\text{Gerretsen}}{\ge}(16R^3-48R^2r-27Rr^2+10r^3)(4R^2+4Rr+3r^2)$$

$$+r(64R^4+176R^3r+108R^2r^2+25Rr^3+2r^4)\overset{?}{\ge}0$$

$$\Leftrightarrow 16t^5-16t^4-19t^3-26t^2-4t+8\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)(16t^4+16t^3+13t^2-4)\overset{?}{\ge}0\rightarrow \text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow (2)\text{ is true}$$

$\therefore$ combining both cases, (2) $\Rightarrow$ (1) is true $\forall\,\Delta$ ABC $\therefore \dfrac{ab}{a^2+b^2}+\dfrac{bc}{b^2+c^2}+\dfrac{ca}{c^2+a^2}$

$\ge\dfrac{1}{2}+\dfrac{2r}{R}\ \forall$ ABC, " = " iff Δ ABC is equilateral (QED)