ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\tan\frac{A}{2}}{\sin B+\sin C}+\frac{\tan\frac{B}{2}}{\sin C+\sin A}+\frac{\tan\frac{C}{2}}{\sin A+\sin B}\ge 1$$

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Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \sin A\ge \sin B\ge \sin C\ \ and\ \tan\frac{A}{2}\ge \tan\frac{B}{2}\ge \tan\frac{C}{2}$$

$$\frac{\tan\frac{A}{2}}{\sin B+\sin C}+\frac{\tan\frac{B}{2}}{\sin C+\sin A}+\frac{\tan\frac{C}{2}}{\sin A+\sin B}\ge$$

$$\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum\tan\frac{A}{2}\right)\left(\sum\frac{1}{\sin B+\sin C}\right)\overset{CBS}{\ge}\frac{1}{3}\cdot\frac{4R+r}{s}\cdot\frac{(1+1+1)^2}{2\sum\sin A}=$$

$$=\frac{1}{3}\cdot\frac{4R+r}{s}\cdot\frac{9R}{2s}=\frac{3R(4R+r)}{2s^2}\overset{Gerretsen}{\ge}\frac{12R^2+3Rr}{2(4R^2+4Rr+3r^2)}=\frac{12R^2+3Rr}{(8R^2+8Rr+6r^2)}$$

$$We\ need\ to\ show,:$$

$$\frac{12R^2+3Rr}{(8R^2+8Rr+6r^2)}\ge 1\ or\ 12R^2+3Rr\ge 8R^2+8Rr+6r^2$$

$$or 4R^2-5Rr-6r^2\ge 0\ or\ (R-2r)(4R+3r)\ge 0\ Euler$$

Equality holds for an equilateral triangle.