ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$r_a^2+r_b^2+r_c^2+l_a^2+l_b^2+l_c^2\ge 2\left(m_a^2+m_b^2+m_c^2\right)$$

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$$\sum_{cyc}\frac{a}{(b+c)^2}=\sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2}=2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2}-\sum_{cyc}\frac{1}{b+c}$$

$$=\frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2-2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2}-\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}$$

$$=\frac{\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}\Rightarrow$$

$$\sum_{cyc}\frac{a}{(b+c)^2}\overset{(\bullet)}{=}\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Now, }\sum_{cyc}w_a^2=\sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2}=\sum_{cyc}\frac{bc\left((b+c)^2-a^2\right)}{(b+c)^2}=\sum_{cyc}\left(bc-\frac{a^2bc}{(b+c)^2}\right)$$

$$\overset{\text{via }(\bullet)}{=}s^2+4Rr+r^2$$

$$+2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\therefore r_b^2+r_c^2+l_a^2+l_b^2+l_c^2\ge 2\left(m_a^2+m_b^2+m_c^2\right)\Leftrightarrow (4R+r)^2-2s^2$$

$$+\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\ge 2.\frac{3}{2}.(s^2-4Rr-r^2)$$

$$\Leftrightarrow -4s^6+(16R^2-3r^2)s^4+r(64R^3+124R^2r+76Rr^2+6r^3)$$

$$+r^2(64R^4+144R^3r+128R^2r^2+44Rr^3+5r^4)\overset{(*)}{\ge}0$$

$$\text{Now, }P=-4s^4(s^2-4R^2-4Rr-3r^2)-(16Rr+15r^2)s^2(s^2-4R^2-4Rr-3r^2)$$

$$-r^2(32Rr+39r^2)(s^2-4R^2-4Rr-3r^2)\overset{\text{Gerretsen}}{\ge}0\ \therefore\text{ in order to prove }(*),$$

it suffices to prove : LHS of $(*)\ge P\Leftrightarrow 16t^4+4t^3-39t^2-52t-28\ge 0\ \left(t=\frac{R}{r}\right)$

$$\Leftrightarrow (t-2)(16t^3+36t^2+33t+14)\ge 0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore r_a^2+r_b^2+r_c^2+l_a^2+l_b^2+l_c^2\ge 2\left(m_a^2+m_b^2+m_c^2\right)\ \forall\ \Delta\ ABC,$$

$''=''$ iff $\Delta$ ABC is equilateral (QED)