ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\tan A + \tan B + \tan C + 6(\sin A + \sin B + \sin C) \geq 12\sqrt{3}$$

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$$\sum_{cyc} \tan A = \prod_{cyc} \frac{\sin A}{\cos A} = \frac{\frac{4Rrs}{8R^3}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}} = \frac{2rs}{s^2-4R^2-4Rr-r^2} \to (1) \text{ and}$$

$$\because f(x) = \tan x \ \forall x \in \left(0, \frac{\pi}{2}\right) \text{ is convex}, \therefore \sum_{cyc} \tan A \, 3\sqrt{3}$$

$$\therefore \frac{2rs}{s^2-4R^2-4Rr-r^2} \geq 3\sqrt{3} \Rightarrow rs \overset{(2)}{\geq} \frac{3\sqrt{3}}{2}(s^2-4R^2-4Rr-r^2) \ \forall \text{ acute } \Delta \text{ ABC}$$

$$\text{Now, } (\text{LHS})^2 \overset{\text{via (1)}}{=} \frac{4r^2s^2}{(s^2-4R^2-4Rr-r^2)^2} + \frac{36s^2}{R^2} + \frac{24rs^2}{R(s^2-4R^2-4Rr-r^2)}$$

$$\overset{?}{\geq} 432 \Leftrightarrow R^2r^2s^2 + (9s^2-108R^2)(s^2-4R^2-4Rr-r^2)^2$$

$$+6Rrs^2(s^2-4R^2-4Rr-r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{Now, via (2), LHS of } (*) \geq \frac{27R^2(s^2-4R^2-4Rr-r^2)^2}{4}$$

$$+(9s^2-108R^2)(s^2-4R^2-4Rr-r^2)^2 + 6Rrs^2(s^2-4R^2-4Rr-r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 12s^4 - (183R^2+40Rr+12r^2)s^2 + 135R^2r^2(2R+r)^2 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and}$$

$$\because 12(s^2-4R^2-4Rr-3r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} 12(s^2-4R^2-4Rr-3r^2)^2$$

$$\Leftrightarrow 348R^4 + 156R^3r - 345R^2r^2 - 288Rr^3 - 108r^4 \underset{(***)}{\overset{?}{\geq}} (87R^2-56Rr-60r^2)s^2$$

$$\text{Now, } (87R^2-56Rr-60r^2)s^2 \overset{\text{Blundon-Gerretsen}}{\leq} (87R^2-56Rr-60r^2).\frac{R(4R+r)^2}{4R-2r}$$

$$\overset{?}{\leq} \text{LHS of } (***) \Leftrightarrow 128t^4 - 371t^3 + 74t^2 + 204t + 216 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(128t^2+141t+126)+144\right) \overset{?}{\geq} 0 \to$$

$$\text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \tan A + \tan B + \tan C + 6(\sin A + \sin B + \sin C) \geq 12\sqrt{3}$$

$$\forall \text{ acute } \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$