ROMANIAN MATHEMATICAL MAGAZINE

In *any acute* ΔABC, the *following relationship holds* :

$$\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\geq 2$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\overset{A-G}{\geq}\sum_{cyc}\frac{4\cos A}{2\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}}=$$

$$=\sum_{cyc}\frac{4\cos A}{1+\cos B+1+\cos C}=4\sum_{cyc}\frac{\cos^2 A}{2\cos A+\cos A\cos B+\cos A\cos C}$$

$$\overset{Bergstrom}{\geq}\frac{4\left(\sum_{cyc}\cos A\right)^2}{2\sum_{cyc}\cos A+2\sum_{cyc}\cos A\cos B}\overset{?}{\geq}2$$

$$\Leftrightarrow 2\left(\sum_{cyc}\cos A\right)^2\overset{?}{\geq}2\sum_{cyc}\cos A+\left(\sum_{cyc}\cos A\right)^2-\left(3-\sum_{cyc}\sin^2 A\right)$$

$$\Leftrightarrow\frac{(R+r)^2}{R^2}-\frac{2(R+r)}{R}+3-\frac{s^2-4Rr-r^2}{2R^2}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{2R^2+4Rr+2r^2-4R^2-4Rr+6R^2-s^2+4Rr+r^2}{2R^2}\overset{?}{\geq}0$$

$$\Leftrightarrow 4R^2+4Rr+3r^2\overset{?}{\geq}s^2\rightarrow \text{true via } a \text{ Gerretsen}$$

$$\therefore\frac{\cos A}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos B}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}}\geq 2\ \forall\ \text{acute } \Delta ABC,$$

$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$