ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\ge\frac{9}{4}$$

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Solution by Tapas Das-India

$$\sum a.m_a^2=\frac{1}{4}\sum a(2b^2+2c^2-a^2)=\frac{1}{4}\left(2\sum a(b^2+c^2)-\sum a^3\right)\ (1)$$

$$2\sum a(b^2+c^2)-\sum a^3=2\sum a^2(b+c)-\sum a^3=$$

$$=2\sum a^2(2s-a)-\sum a^3=4s\sum a^2-2\sum a^3-\sum a^3=$$

$$=8s(s^2-r^2-4Rr)-6s(s^2-3r^2-6Rr)=$$

$$=8s^3-8s(4Rr+r^2)-6s^3+6s(6Rr+3r^2)=$$

$$=2s^3-2s(16Rr+4r^2-9r^2-18Rr)=$$

$$=2s(s^2+2Rr+5r^2)\overset{Gerretsen}{\ge}2s(16Rr-5r^2+2Rr+5r^2)=36Rrs\ (2)$$

$$\sum am_a^2=\frac{1}{4}\sum a(2b^2+2c^2-a^2)=\frac{1}{4}\left(2\sum a(b^2+c^2)-\sum a^3\right)\overset{(2)}{\ge}$$

$$\ge\frac{1}{4}.36Rrs=9Rrs$$

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}=\sum\frac{m_a^2}{bc}=\frac{1}{abc}\sum am_a^2\ge\frac{1}{abc}.9Rrs=\frac{9Rrs}{4Rrs}=\frac{9}{4}$$

Equality holds for an equilateral triangle