ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, $\forall x,y,z > 0$, the following relationship holds :

$$\frac{x}{y+z}(1+\cos A)+\frac{y}{z+x}(1+\cos B)+\frac{z}{x+y}(1+\cos C)\ge\frac{\sqrt{3}}{2}(\sin A+\sin B+\sin C)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=\frac{s(s-a)}{bc}+\frac{s(s-b)}{ca}-\frac{s(s-c)}{ab}=$$

$$=\frac{s}{abc}\big(a(s-a)+b(s-b)-c(s-c)\big)=\frac{s}{abc}\big(x(y+z)+y(z+x)-z(x+y)\big)=$$

$$=\frac{s}{abc}(2xy)>0(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\therefore\cos^2\frac{C}{2}<\cos^2\frac{A}{2}+\cos^2\frac{B}{2}<\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^2\Rightarrow\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}>0$$

and analogs $\Rightarrow\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}$ form sides of a triangle

$\Rightarrow\sqrt{\cos\frac{A}{2}},\sqrt{\cos\frac{B}{2}},\sqrt{\cos\frac{C}{2}}$ form sides of a triangle with area

$$F_1=\frac{1}{4}\cdot\sqrt{2\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}-\sum_{cyc}\cos^2\frac{A}{2}}$$

$$\text{Now, }\frac{x}{y+z}(1+\cos A)+\frac{y}{z+x}(1+\cos B)+\frac{z}{x+y}(1+\cos C)$$

$$=2\left(\frac{x}{y+z}\cdot\left(\sqrt{\cos\frac{A}{2}}\right)^4+\frac{y}{z+x}\cdot\left(\sqrt{\cos\frac{B}{2}}\right)^4+\frac{z}{x+y}\cdot\left(\sqrt{\cos\frac{C}{2}}\right)^4\right)\ge$$

$$\overset{\text{Tsintsifas}}{\ge}16F_1^2=2\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}-\sum_{cyc}\cos^2\frac{A}{2}\overset{?}{\ge}\frac{\sqrt{3}}{2}(\sin A+\sin B+\sin C)$$

$$\Leftrightarrow 2\sum_{cyc}\sin\frac{B+C}{2}\sin\frac{C+A}{2}-\sum_{cyc}\sin^2\frac{B+C}{2}\underset{(1)}{\overset{?}{\ge}}\frac{\sqrt{3}}{2}\left(\sum_{cyc}\sin(B+C)\right)$$

$$\text{Now, since : }\frac{B+C}{2}+\frac{C+A}{2}+\frac{A+B}{2}=\pi,\therefore X=\frac{B+C}{2},Y=\frac{C+A}{2},Z=\frac{A+B}{2}$$

form angles of a triangle XYZ with sides x,y,z and semiperimeter, circumradius, inradius $=s',R'$ and r' (say) and then

$$:(1)\Leftrightarrow 2\sum_{cyc}\sin X\sin Y-\sum_{cyc}\sin^2X\overset{?}{\ge}\frac{\sqrt{3}}{2}\cdot\sum_{cyc}\sin 2X$$

$$\Leftrightarrow \frac{1}{4R'^2}\left(2\sum_{cyc} xy - \sum_{cyc} x^2\right) \overset{?}{\geq} \frac{\sqrt{3}}{2}.4\sin X\sin Y\sin Z = 2\sqrt{3}.\frac{xyz}{8R'^3} = 2\sqrt{3}.\frac{4R'r's'}{8R'^3}$$

$$\Leftrightarrow 2\sum_{cyc} xy - \sum_{cyc} x^2 \overset{?}{\geq} 4\sqrt{3}.r's' \rightarrow \text{which is true via Hadwiger} - \text{Finsler}$$

$$\Rightarrow \text{①} \text{ is true} \therefore \frac{x}{y+z}(1+\cos A) + \frac{y}{z+x}(1+\cos B) + \frac{z}{x+y}(1+\cos C)$$

$$\geq \frac{\sqrt{3}}{2}(\sin A + \sin B + \sin C)\ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$