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In $\triangle ABC$ the following relationship holds:

$$\frac{\cos \frac{A}{2}}{1 + \cos A} + \frac{\cos \frac{B}{2}}{1 + \cos B} + \frac{\cos \frac{C}{2}}{1 + \cos C} \geq \sqrt{3}$$

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Solution by Daniel Sitaru-Romania

$$\begin{aligned} \sum_{cyc} \frac{\cos \frac{A}{2}}{1 + \cos A} &= \sum_{cyc} \frac{\cos \frac{A}{2}}{1 + 2\cos^2 \frac{A}{2} - 1} = \frac{1}{2} \sum_{cyc} \frac{1}{\cos \frac{A}{2}} = \\ &= \frac{1}{2} \sum_{cyc} \frac{1^2}{\cos \frac{A}{2}} \stackrel{\text{BERGSTROM}}{\geq} \frac{1}{2} \cdot \frac{(1 + 1 + 1)^2}{\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}} \stackrel{\text{JENSEN}}{\geq} \\ &\geq \frac{9}{2} \cdot \frac{1}{3\cos\left(\frac{A+B+C}{6}\right)} = \frac{3}{2\cos\frac{\pi}{6}} = \frac{3}{2 \cdot \frac{\sqrt{3}}{2}} = \sqrt{3} \end{aligned}$$

Equality holds for $a = b = c$.