ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{9R}{a^2+b^2+c^2}\le\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}\le\frac{1}{2r}$$

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Solution by Tapas Das-India

$$\frac{1}{h_a+\sqrt{h_bh_c}}\overset{AM-HM}{\le}\frac{1}{4}\left(\frac{1}{h_a}+\frac{1}{\sqrt{h_bh_c}}\right)=$$

$$=\frac{1}{4}\left(\frac{1}{h_a}+\sqrt{\frac{1}{h_b}\cdot\frac{1}{h_c}}\right)\overset{AM-GM}{\le}\frac{1}{4}\left(\frac{1}{h_a}+\frac{1}{2}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)\ (1)$$

$$\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}=\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{(1)}{\le}$$

$$\le\frac{1}{4}\sum\left(\frac{1}{h_a}+\frac{1}{2}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\right)=\frac{2}{4}\sum\frac{1}{h_a}=\frac{1}{2r}$$

$$h_a+h_b+h_c=2F\frac{ab+bc+ca}{abc}=\frac{2F}{4RF}\sum ab\le\frac{1}{2R}(a^2+b^2+c^2)\ (2)$$

$$\frac{1}{h_a+\sqrt{h_bh_c}}+\frac{1}{h_b+\sqrt{h_ch_a}}+\frac{1}{h_c+\sqrt{h_ah_b}}=\sum\frac{1}{h_a+\sqrt{h_bh_c}}\overset{AM-GM}{\ge}$$

$$\ge\sum\frac{1}{h_a+\frac{h_b+h_c}{2}}\overset{CBS}{\ge}\frac{(1+1+1)^2}{2(h_a+h_b+h_c)}\overset{(2)}{\ge}\frac{9R}{a^2+b^2+c^2}$$

$$\textit{Equality holds for } a=b=c$$