ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$\tan\frac{C}{2}(a^2\tan A+b^2\tan B)\geq a^2+b^2$$

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WLOG: $a\leq b\rightarrow \tan A\leq \tan B$

$$\tan\frac{C}{2}(a^2\tan A+b^2\tan B)\overset{CEBYSHEV}{\geq}$$

$$\geq\frac{1}{2}\tan\frac{C}{2}(a^2+b^2)(\tan A+\tan B)\geq a^2+b^2\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{2}\tan\frac{C}{2}(\tan A+\tan B)\geq 1\Leftrightarrow$$

$$\Leftrightarrow\tan\frac{C}{2}\cdot\frac{\sin(A+B)}{\cos A\cos B}\geq 2\Leftrightarrow\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\cdot\sin(\pi-C)\geq 2\cos A\cos B\Leftrightarrow$$

$$\Leftrightarrow\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\cdot 2\sin\frac{C}{2}\cos\frac{C}{2}\geq 2\cos A\cos B\Leftrightarrow$$

$$\Leftrightarrow 2\sin^2\frac{C}{2}\geq\ 2\cos A\cos B\Leftrightarrow 1-\cos C\geq\ 2\cos A\cos B\Leftrightarrow$$

$$\Leftrightarrow 1+\cos(A+B)\geq\ 2\cos A\cos B\Leftrightarrow$$

$$2\cos A\cos B-\cos A\cos B+\sin A\sin B\leq 1\Leftrightarrow\cos(A-B)\leq 1$$

Equality holds for $A=B$.