ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$r_a cosA + r_b cosB + r_c cosC \leq \frac{9R}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a \leq b \leq c \Rightarrow -a \geq -b \geq -c \Rightarrow s-a \geq s-b \geq s-c \Rightarrow$

$$\frac{1}{s-a} \leq \frac{1}{s-b} \leq \frac{1}{s-c} \Rightarrow \frac{F}{s-a} \leq \frac{F}{s-b} \leq \frac{F}{s-c} \Rightarrow r_a \leq r_b \leq r_c$$

$$a \leq b \leq c \Rightarrow cosA \geq cosB \geq cosC$$

$$\sum_{cyc} r_a cosA \overset{CEBYSHEV}{\leq} \frac{1}{3} \cdot \sum_{cyc} r_a \cdot \sum_{cyc} cosA \overset{KLAMKIN}{\leq} \frac{1}{3} \cdot \frac{9R}{2} \cdot \sum_{cyc} cosA =$$

$$= \frac{3R}{2} \cdot \left(1 + \frac{r}{R}\right) \overset{EULER}{\leq} \frac{3R}{2} \cdot \left(1 + \frac{1}{2}\right) = \frac{9R}{4}$$

Equality holds for $a = b = c$.