

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$((a^2 + b^2)\cos C + (b^2 + c^2)\cos A + (c^2 + a^2)\cos B) \leq 9R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\text{WLOG: } a \leq b \leq c \rightarrow a^2 \leq b^2 \leq c^2 \rightarrow \begin{cases} a^2 + b^2 \leq c^2 + b^2 \\ a^2 + c^2 \leq b^2 + c^2 \\ a^2 + b^2 \leq a^2 + c^2 \end{cases} \rightarrow$$

$$\rightarrow a^2 + b^2 \leq a^2 + c^2 \leq b^2 + c^2$$

$$a \leq b \leq c \rightarrow \cos A \geq \cos B \geq \cos C$$

$$\begin{aligned} \sum_{cyc} (a^2 + b^2) \cos C &\leq \frac{1}{3} \sum_{cyc} (a^2 + b^2) \cdot \sum_{cyc} \cos C = \\ &= \frac{2}{3} \sum_{cyc} a^2 \cdot \left(1 + \frac{r}{R}\right) \stackrel{EULER}{\leq} \frac{2}{3} \cdot 2(s^2 - r^2 - 4Rr) \left(1 + \frac{1}{2}\right) \leq \end{aligned}$$

$$\stackrel{GERRETSEN}{\leq} 2(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) = 2(4R^2 + 2r^2) \leq$$

$$\stackrel{EULER}{\leq} 2 \left(4R^2 + \frac{R^2}{2}\right) = 9R^2$$

Equality holds for: $a = b = c$.