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In any Δ ABC, the following relationship holds :

$$\frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}\ge\frac{r(5R-2r)(4R+r)}{4R^2}\cdot\sqrt{\frac{2(5R-2r)}{2R-r}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}=\sum_{cyc}\frac{2rs}{a\cos\frac{A}{2}}\overset{\text{Bergstrom}}{\ge}\frac{18rs}{\sum_{cyc}a\cos\frac{A}{2}}$$

$$=\frac{18rs}{\sum_{cyc}\left(a.\sqrt{\frac{sa(s-a)}{4Rrs}}\right)}=\frac{36rs.\sqrt{Rr}}{\sum_{cyc}\left(\sqrt{a}.\sqrt{a^2(s-a)}\right)}$$

$$\overset{\text{CBS}}{\ge}\frac{36rs.\sqrt{Rr}}{\sqrt{2s}.\sqrt{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}}=\frac{18r.\sqrt{Rr}}{\sqrt{2Rr+2r^2}}=\frac{9r.\sqrt{2R}}{\sqrt{R+r}}$$

$$\overset{?}{\ge}\frac{r(5R-2r)(4R+r)}{4R^2}\cdot\sqrt{\frac{2(5R-2r)}{2R-r}}$$

$$\Leftrightarrow 1296R^5(2R-r)\overset{?}{\ge}(R+r)(4R+r)^2(5R-2r)^3$$

$$\Leftrightarrow 592t^6-1896t^5+1515t^4-87t^3-198t^2+12t+8\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(\begin{array}{c}236t^5+356t^4(t-2)+91t^3++90t^2+4t(t-2)\\+(t-2)(t+2)\end{array}\right)\overset{?}{\ge}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2\ \therefore\frac{h_a}{\cos\frac{A}{2}}+\frac{h_b}{\cos\frac{B}{2}}+\frac{h_c}{\cos\frac{C}{2}}\ge\frac{r(5R-2r)(4R+r)}{4R^2}\cdot\sqrt{\frac{2(5R-2r)}{2R-r}}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)