ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\geq 2\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}$$

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$$\sum_{cyc}\left(a\sin\frac{A}{2}\right)=4R\sum_{cyc}\left(\sin^2\frac{A}{2}\cos\frac{A}{2}\right)=R\sum_{cyc}\left(2\sin^2\frac{A}{2}.\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)\geq$$

$$\overset{0<\cos\frac{B-C}{2}\leq 1\ and\ analogs}{\geq} R\sum_{cyc}\left((1-\cos A)\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right)\right)=$$

$$=R\sum_{cyc}\left((1-\cos A)(\sin B+\sin C)\right)=$$

$$=R\sum_{cyc}(\sin B+\sin C)-R\sum_{.cyc}\left(\cos A\left(\sum_{cyc}\sin A-\sin A\right)\right)$$

$$=2R\cdot\frac{s}{R}-R.\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\sin A\right)+\frac{R}{2}.\sum_{cyc}\sin 2A=$$

$$=2s-R\left(\frac{R+r}{R}\right)\left(\frac{s}{R}\right)+2R.\frac{4Rrs}{8R^3}=2s-s-\frac{rs}{R}+\frac{rs}{R}\Rightarrow\sum_{cyc}\left(a\sin\frac{A}{2}\right)\geq s\ and\ so,$$

$$\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\overset{Panaitopol}{\geq}\sum_{cyc}\frac{a\sin\frac{A}{2}}{Rs}\geq\frac{s}{Rs}=\frac{1}{R}\overset{?}{\geq}2.\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}$$

$$\Leftrightarrow R^2-4Rr+4r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)^2\overset{?}{\geq}0\to$$

$$true\ \therefore\frac{\sin\frac{A}{2}}{m_a}+\frac{\sin\frac{B}{2}}{m_b}+\frac{\sin\frac{C}{2}}{m_c}\geq 2\sqrt{\frac{2R-r}{R(9R^2-8Rr+4r^2)}}\ \forall\Delta ABC$$

" = " iff $\Delta$ ABC is equilateral (QED)