In $any\ \Delta\ ABC$, the following relationship holds :

$$\frac{w_a w_b}{\sqrt{r_c}}+\frac{w_b w_c}{\sqrt{r_a}}+\frac{w_c w_a}{\sqrt{r_b}}\ge 4\sqrt{2}r(r+4R)^2.\sqrt{\frac{R}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}}$$

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$$\frac{w_a w_b}{\sqrt{r_c}}+\frac{w_b w_c}{\sqrt{r_a}}+\frac{w_c w_a}{\sqrt{r_b}}=w_a w_b w_c.\sum_{cyc}\frac{\sqrt{s-a}.(b+c)}{\sqrt{rs}.2\sqrt{bc}.\sqrt{s(s-a)}}=$$

$$\frac{8Rr^2s}{s^2+2Rr+r^2}.\frac{1}{\sqrt{4Rr^2s}}.\sum_{cyc}\frac{(b+c)\sqrt{a}.a.\sqrt{b+c}}{\sqrt{a^2(b+c)}}=\frac{4r.\sqrt{Rs}}{s^2+2Rr+r^2}.\sum_{cyc}\frac{\left(a(b+c)\right)^{\frac{3}{2}}}{\sqrt{a^2(b+c)}}$$

$$\overset{Radon}{\ge}\frac{4r.\sqrt{Rs}}{s^2+2Rr+r^2}.\frac{\left(2(s^2+4Rr+r^2)\right)^{\frac{3}{2}}}{\sqrt{2s(s^2+4Rr+r^2)-12Rrs}}\overset{?}{\ge}$$

$$4\sqrt{2}r(r+4R)^2.\sqrt{\frac{R}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}}$$

$$\Leftrightarrow\frac{8Rs(s^2+4Rr+r^2)^3}{(s^2+2Rr+r^2)^2.2s(s^2-2Rr+r^2)}\overset{?}{\ge}\frac{2R(r+4R)^4}{128R^4+32R^3r-117R^2r^2-52Rr^3-4r^4}$$

$$\Leftrightarrow -(192R^3+330R^2r+120Rr^2+9r^3)s^6+$$
$$(2560R^5+256R^4r-3576R^3r^2-2270R^2r^3-458Rr^4-27r^5)s^4$$
$$+r\begin{pmatrix}13312R^6+9216R^5r-10336R^4r^2-11504R^3r^3-3930R^2r^4\\-556Rr^5-27r^6\end{pmatrix}s^2+$$
$$r^2\begin{pmatrix}18432R^7+19456R^6r-7552R^5r^2-17120R^4r^3-8624R^3r^4\\-1990R^2r^5-218Rr^6-9r^7\end{pmatrix}\overset{?}{\ge}0\quad (1)$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(*)}{\le}0$$

$$\Rightarrow -(192R^3+330R^2r+120Rr^2+9r^3)s^2\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}\ge 0$$

and so, in order to prove (1), it suffices to prove : LHS of (1) $\ge$

$$-(192R^3+330R^2r+120Rr^2+9r^3)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow(1792R^5-4904R^4r-10272R^3r^2-4046R^2r^3-398Rr^4-9r^5)s^4+$$
$$r\begin{pmatrix}25600R^6+39552R^5r+15488R^4r^2-1016R^3r^3-1728R^2r^4\\-328Rr^5-18r^6\end{pmatrix}s^2+$$
$$r^2\left((R-2r)\begin{pmatrix}18432R^6+56320R^5r+105088R^4r^2+193056R^3r^3\\+377488R^2r^4+752986Rr^5+1505754r^6\\+3011499r^7\end{pmatrix}\right)\overset{(2)}{\ge}0$$

and it's trivially true if :

$$1792R^5 - 4904R^4r - 10272R^3r^2 - 4046R^2r^3 - 398Rr^4 - 9r^5 \geq 0 \left(\because R \overset{\text{Euler}}{\geq} 2r\right)$$

and so, we now focus on the case when :

$$1792R^5 - 4904R^4r - 10272R^3r^2 - 4046R^2r^3 - 398Rr^4 - 9r^5 < 0 \text{ and then :}$$

$$\begin{pmatrix} 1792R^5 - 4904R^4r - 10272R^3r^2 \\ -4046R^2r^3 - 398Rr^4 - 9r^5 \end{pmatrix} \begin{pmatrix} s^4 - s^2(4R^2 + 20Rr - 2r^2) \\ +r(4R+r)^3 \end{pmatrix} \overset{\text{via }(*)}{\geq} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\geq 0 \Leftrightarrow$

$$(896R^6 + 5228R^5r - 12900R^4r^2 - 24541R^3r^3 - 7873R^2r^4 - 204Rr^5 + 36r^6)s^2$$

$$\overset{③}{\geq} r\begin{pmatrix} 14336R^7 - 30784R^6r - 111344R^5r^2 - 100188R^4r^3 - 41341R^3r^4 \\ -8735R^2r^5 - 908Rr^6 - 36r^7 \end{pmatrix}$$

Case 1 $896R^6 + 5228R^5r - 12900R^4r^2 - 24541R^3r^3 - 7873R^2r^4 - 204Rr^5$

$+36r^6 \geq 0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$\begin{pmatrix} 896R^6 + 5228R^5r - 12900R^4r^2 - 24541R^3r^3 - 7873R^2r^4 \\ -204Rr^5 + 36r^6 \end{pmatrix}(16Rr - 5r^2)$$

$\overset{?}{\geq}$ RHS of ③ $\Leftrightarrow 54976t^6 - 60598t^5 - 113984t^4 + 19039t^3 + 22418t^2$

$$+1252t - 72 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\begin{pmatrix} (t-2)(54976t^4 + 159306t^3 + 303336t^2 + 595159t + 1189710) \\ +2379456 \end{pmatrix}$$

$\overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ③ is true

Case 2 $896R^6 + 5228R^5r - 12900R^4r^2 - 24541R^3r^3 - 7873R^2r^4$

$-204Rr^5 + 36r^6 < 0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$\begin{pmatrix} 896R^6 + 5228R^5r - 12900R^4r^2 - 24541R^3r^3 - 7873R^2r^4 \\ -204Rr^5 + 36r^6 \end{pmatrix}(4R^2 + 4Rr + 3r^2)$$

$\overset{?}{\geq}$ RHS of ③ $\Leftrightarrow 1792t^8 + 5080t^7 + 1392t^6 - 11368t^5 - 34084t^4 - 32295t^3$

$$-7778t^2 + 220t + 72 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\begin{pmatrix} 1792t^7 + 8664t^6 + 18720t^5 + 26072t^4 + 18060t^3 + 3825t^2 \\ -128t - 36 \end{pmatrix} \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\forall \Delta ABC \therefore \frac{w_a w_b}{\sqrt{r_c}} + \frac{w_b w_c}{\sqrt{r_a}} + \frac{w_c w_a}{\sqrt{r_b}} \geq$$

$$4\sqrt{2}r(r+4R)^2 \cdot \sqrt{\frac{R}{128R^4 + 32R^3r - 117R^2r^2 - 52Rr^3 - 4r^4}} \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)