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In any Δ ABC, the following relationship holds :

$$\frac{h_a}{w_a\sqrt{r_a}}+\frac{h_b}{w_b\sqrt{r_b}}+\frac{h_c}{w_c\sqrt{r_c}}\ge 4\sqrt{\frac{2R-r}{R(5R-2r)}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{w_a\sqrt{r_a}}=\frac{bc.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2R.2bc\cos\frac{A}{2}}\cdot\frac{\sqrt{s-a}}{\sqrt{rs}}=\frac{b+c}{a}\sin\frac{A}{2}\cdot\frac{\sqrt{s-a}}{\sqrt{rs}}$$

$$=\frac{b+c}{abc}\cdot\frac{bc}{\sqrt{bc}}\cdot\sqrt{\frac{(s-a)(s-b)(s-c)}{rs}}=\frac{b+c}{4Rrs}\cdot\sqrt{bc}.\sqrt{r}$$

$$\Rightarrow\frac{h_a}{w_a\sqrt{r_a}}=\frac{\sqrt{r}}{4Rrs}\cdot\sqrt{bc}(b+c)\text{ and analogs}\Rightarrow\left(\sum_{cyc}\frac{h_a}{w_a\sqrt{r_a}}\right)^2=$$

$$\frac{1}{16R^2rs^2}\cdot\left(\sum_{cyc}\left(bc(b^2+c^2+a^2+2bc-a^2)\right)+2.\sum_{cyc}\left(\sqrt{bc}.\sqrt{ca}.(b+c)(c+a)\right)\right)$$

$$\overset{GM-HM}{\ge}\frac{1}{16R^2rs^2}\cdot\left(\begin{matrix}2(s^2-4Rr-r^2)(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^2-32Rrs^2\\-8Rrs^2+8\sum_{cyc}\left(\frac{bc}{b+c}\cdot\frac{ca}{c+a}.(b+c)(c+a)\right)\end{matrix}\right)$$

$$=\frac{1}{16R^2rs^2}\cdot\left(\begin{matrix}2(s^2-4Rr-r^2)(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^2-32Rrs^2\\-8Rrs^2+8(4Rrs)(2s)\end{matrix}\right)$$

$$=\frac{s^2+10Rr+r^2}{4R^2r}\overset{Gerretsen}{\ge}\frac{13R-2r}{2R^2}\overset{?}{\ge}16.\frac{2R-r}{R(5R-2r)}\Leftrightarrow R^2-4Rr+4r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)^2\overset{?}{\ge}0\to\text{true}\therefore\frac{h_a}{w_a\sqrt{r_a}}+\frac{h_b}{w_b\sqrt{r_b}}+\frac{h_c}{w_c\sqrt{r_c}}\ge 4.\sqrt{\frac{2R-r}{R(5R-2r)}}\ \forall\Delta ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{h_a}{w_a\sqrt{r_a}}=\sum_{cyc}\frac{(b+c)\sqrt{r}}{a\sqrt{bc}}=\sqrt{\frac{r}{abc}}\cdot\sum_{cyc}\frac{b+c}{\sqrt{a}}\overset{H\ddot{o}lder}{\ge}\sqrt{\frac{1}{4Rs}\cdot\frac{\left(\sum_{cyc}(b+c)\right)^3}{\sum_{cyc}a(b+c)}}=$$

$$=\frac{4s}{\sqrt{2R(s^2+r^2+4Rr)}}\overset{?}{\geq}4\sqrt{\frac{2R-r}{R(5R-2r)}}\Leftrightarrow s^2\geq\frac{2r(2R-r)(4R+r)}{R},$$

which is Gerretsen − Blundon inequality.

So the proof is complete. Equality holds iff ΔABC is equilateral.