ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}\ge\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}$$

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$$\text{Let } f(x)=\sqrt{\tan\frac{x}{2}}.\sin\frac{x}{2}\ \forall x\in(0,\pi) \text{ and then : } f''(x)=$$

$$\frac{\left(4\sec^2\frac{x}{2}-4\right)\left(\sin\frac{x}{2}\right)\left(\tan^2\frac{x}{2}\right)+4\left(\sec^2\frac{x}{2}\right)\left(\sin\frac{x}{2}\right)-\left(\sin\frac{x}{2}\right)\left(\sec^4\frac{x}{2}\right)}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(4\left(\sec^4\frac{x}{2}-2\sec^2\frac{x}{2}+1\right)+4\sec^2\frac{x}{2}-\sec^4\frac{x}{2}\right)$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(3\sec^4\frac{x}{2}-4\sec^2\frac{x}{2}+4\right)$$

$$=\frac{\sin\frac{x}{2}}{16\left(\tan\frac{x}{2}\right)^{\frac{3}{2}}}.\left(2\sec^4\frac{x}{2}+\left(\sec^2\frac{x}{2}-2\right)^2\right)>0\Rightarrow f(x)=\sqrt{\tan\frac{x}{2}}.\sin\frac{x}{2}$$

$$\forall x\in(0,\pi)\text{ is convex}$$

$$\therefore\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}=$$

$$=\sqrt{s}.\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}}.\sin\frac{A}{2}\right)\overset{\text{Jensen}}{\ge}3\sqrt{s}.\sqrt{\tan\frac{\pi}{6}}.\sin\frac{\pi}{6}=\frac{3\sqrt{s}}{2}.\sqrt{\frac{1}{\sqrt{3}}}\overset{?}{\ge}$$

$$\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}\Leftrightarrow\frac{9R}{4\sqrt{3}}\underset{(1)}{\overset{?}{\ge}}\frac{s^3}{((4R+r)^2-s^2)}$$

$$\text{Now, LHS of }(1)\overset{\text{Mitrinovic}}{\ge}\frac{9.2s}{(4\sqrt{3})(3\sqrt{3})}=\frac{s}{2}\overset{?}{\ge}\frac{s^3}{((4R+r)^2-s^2)}$$

$$\Leftrightarrow(4R+r)^2-s^2\overset{?}{\ge}2s^2\Leftrightarrow(4R+r)^2\overset{?}{\ge}3s^2\rightarrow\text{true via Doucet}\Rightarrow(1)\text{ is true}$$

$$\therefore\sqrt{r_a}\sin\frac{A}{2}+\sqrt{r_b}\sin\frac{B}{2}+\sqrt{r_c}\sin\frac{C}{2}\ge\frac{s^2}{\sqrt{R(4R+r+s)(4R+r-s)}}\ \forall\Delta\text{ ABC},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$