ROMANIAN MATHEMATICAL MAGAZINE

For a non − obtuse ΔABC, prove that

$$2(\tan A+\tan B)+\tan C\geq\frac{1}{4}\sqrt{5+\sqrt{17}}\left(\sqrt{17}+7\right)$$

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Since $x\to\tan x$ is convex on $\left(0,\frac{\pi}{2}\right)$, then by Jensen's inequality, we have

$$2(\tan A+\tan B)+\tan C\geq 4\tan\left(\frac{A+B}{2}\right)+\tan C=\frac{4}{\tan\frac{C}{2}}+\frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}}=f\left(\tan\frac{C}{2}\right),$$

where $f(x)=\frac{4}{x}+\frac{2x}{1-x^2}, x\in(0,1)$. We have

$$f'(x)=-\frac{4}{x^2}+\frac{2(1+x^2)}{(1-x^2)^2}=\frac{2(-2+5x^2-x^4)}{(1-x^2)^2x^2}=\frac{2\left(x^2-\frac{5-\sqrt{17}}{2}\right)\left(\frac{5+\sqrt{17}}{2}-x^2\right)}{(1-x^2)^2x^2},$$

then $\min\limits_{x\in(0,1)} f(x)=f\left(\sqrt{\frac{5-\sqrt{17}}{2}}\right)=\frac{1}{4}\sqrt{5+\sqrt{17}}\left(\sqrt{17}+7\right).$

Therefore

$$2(\tan A+\tan B)+\tan C\geq\frac{1}{4}\sqrt{5+\sqrt{17}}\left(\sqrt{17}+7\right).$$

Equality holds iff $C=2\tan^{-1}\sqrt{\frac{5-\sqrt{17}}{2}}, A=B=\frac{\pi-C}{2}.$