ROMANIAN MATHEMATICAL MAGAZINE

Given triangle ABC such that $A = 2B$, prove that :

$$\frac{109}{64} \leq 5\cos^2 A + 5\sin^2 B - \cos C < 6$$

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We have $C = \pi - A - B = \pi - 3B > 0$, then $B < \frac{\pi}{3}$. Let $x := \cos B \in \left(\frac{1}{2}, 1\right)$, we have

$$5\cos^2 A + 5\sin^2 B - \cos C = 5\cos^2 2B + 5\sin^2 B + \cos 3B$$

$$= 5\left(2x^2 - 1\right)^2 + 5\left(1 - x^2\right) + 4x^3 - 3x = 10 - 3x - 25x^2 + 4x^3 + 20x^4$$

$$= 6 - (1 - x)(2x + 1)\left(10x^2 + 7x - 4\right) < 6.$$

$$5\cos^2 A + 5\sin^2 B - \cos C = 10 - 3x - 25x^2 + 4x^3 + 20x^4$$

$$= \frac{109}{64} + \frac{531}{64} - 3x - 25x^2 + 4x^3 + 20x^4 = \frac{109}{64} + \left(x - \frac{3}{4}\right)^2 \left(20x^2 + 34x + \frac{59}{4}\right) \geq \frac{109}{64},$$

with equality when $x = \frac{3}{4}$ or $B = \cos^{-1}\left(\frac{3}{4}\right), A = 2\cos^{-1}\left(\frac{3}{4}\right), C = \pi - 3\cos^{-1}\left(\frac{3}{4}\right)$.

Therefore

$$\frac{109}{64} \leq 5\cos^2 A + 5\sin^2 B - \cos C < 6$$