ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds:

$$\frac{a^2+b^2+c^2}{2R} \le \frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a \le \frac{9R}{2}$$

Proposed by Tapas Das-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a = \sum_{cyc}\frac{s_a}{h_a}m_b = \sum_{cyc}\frac{2bcm_a}{b^2+c^2}\cdot\frac{a}{2F}\cdot m_b = 4R\sum_{cyc}\frac{m_am_b}{b^2+c^2} \ge$$

$$\overset{Tereshin}{\ge} 4R\sum_{cyc}\frac{(b^2+c^2)(c^2+a^2)}{16R^2(b^2+c^2)} = \frac{a^2+b^2+c^2}{2R}$$

$$\bullet \frac{s_a}{h_a}m_b + \frac{s_b}{h_b}m_c + \frac{s_c}{h_c}m_a = 4R\sum_{cyc}\frac{m_am_b}{b^2+c^2} = \frac{R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}\frac{4\sqrt{a^2+b^2+c^2}m_a}{b^2+c^2}m_b$$

$$\overset{AM-GM}{\le} \frac{R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}\frac{a^2+b^2+c^2+4{m_a}^2}{b^2+c^2}m_b =$$

$$= \frac{3R}{\sqrt{a^2+b^2+c^2}}\sum_{cyc}m_b \overset{CBS}{\le} \frac{3R\sqrt{3({m_a}^2+{m_a}^2+{m_a}^2)}}{\sqrt{a^2+b^2+c^2}} = \frac{9R}{2}.$$

Equality holds iff ΔABC is equilateral.