ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds:

$$a(m_b w_b + m_c w_c) + b(m_c w_c + m_a w_a) + c(m_a w_a + m_b w_b) \geq \frac{2s^3}{\sqrt{2 + \frac{r}{2R}}}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using the known inequality $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, we have:

$m_a w_a \geq s(s-a)$ (and analogs), then:

$$\sum_{cyc} a(m_b w_b + m_c w_c) \geq \sum_{cyc} a\big(s(s-b) + s(s-c)\big) = s\sum_{cyc} a^2 = 2s(s^2 - 4Rr - r^2) =$$

$$= 2s^3\left(1 - \frac{4Rr + r^2}{s^2}\right) \overset{Gerretsen}{\geq} 2s^3\left(1 - \frac{4Rr + r^2}{16Rr - 5r^2}\right) = \frac{2s^3(12R - 6r)}{16R - 5r} \overset{?}{\geq} \frac{2s^3}{\sqrt{2 + \frac{r}{2R}}}$$

$$\Leftrightarrow 6(2R - r)\sqrt{4R + r} \geq (16R - 5r)\sqrt{2R} \overset{squaring}{\Leftrightarrow} (R - 2r)\left(64R^2 + 16Rr - 18r^2\right) \geq 0,$$

which is true and the proof is complete. Equality holds iff ΔABC is equilateral.