ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{m_a}{(m_b+m_c)^2}\geq\frac{\sum_{cyc}\left((b^2+c^2)\sin 2A\right)}{\sum_{cyc}(a^3\cos(B-C))}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(a^3\cos(B-C))=R\sum_{cyc}(a^2.2\sin(B+C)\cos(B-C))$$

$$=R\sum_{cyc}\left(a^2\left(\sum_{cyc}\sin 2A-\sin 2A\right)\right)$$

$$\Rightarrow\sum_{cyc}(a^3\cos(B-C))=R\left(\sum_{cyc}\sin 2A\right)\left(\sum_{cyc}a^2\right)-R\sum_{cyc}(a^2.\sin 2A)\rightarrow(1)\text{ and}$$

$$\sum_{cyc}\left((b^2+c^2)\sin 2A\right)=\left(\sum_{cyc}\sin 2A\right)\left(\sum_{cyc}a^2\right)-\sum_{cyc}(a^2.\sin 2A)\rightarrow(2)$$

$$\text{and }2\sum_{cyc}\frac{m_a}{(m_b+m_c)^2}=2\sum_{cyc}\frac{m_a^3}{(m_am_b+m_am_c)^2}\overset{\text{Radon}}{\geq}\frac{2\left(\sum_{cyc}m_a\right)^3}{\frac{4}{9}\left(3\sum_{cyc}m_am_b\right)^2}$$

$$\geq\frac{\left(\sum_{cyc}m_a\right)^3}{\frac{2}{9}\left(\sum_{cyc}m_a\right)^4}\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{Euler}}}{\geq}\frac{1}{\frac{2}{9}.\frac{9R}{2}}=\frac{1}{R}\overset{\text{via }(1),(2)}{=}\frac{\sum_{cyc}\left((b^2+c^2)\sin 2A\right)}{\sum_{cyc}(a^3\cos(B-C))}\ \forall\,\Delta\text{ ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)