ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{(b^3+c^3)\cot\frac{B}{2}.\frac{\sin C}{\sin A}+(c^3+a^3)\cot\frac{C}{2}.\frac{\sin B}{\sin A}}{\sin B+\sin C}\ge 288\sqrt{3}r^3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$LHS=2R(b^3+c^3)\cot\frac{B}{2}\left(\frac{\frac{c}{a}}{b+c}+\frac{\frac{a}{c}}{a+b}\right)+$$

$$+2R(c^3+a^3)\cot\frac{C}{2}\left(\frac{\frac{b}{a}}{b+c}+\frac{\frac{a}{b}}{a+c}\right)+2R(a^3+b^3)\cot\frac{A}{2}\left(\frac{\frac{c}{b}}{a+c}+\frac{\frac{b}{c}}{a+b}\right)\ge$$

$$\overset{A-G}{\ge}\frac{4R(b^3+c^3)\cot\frac{B}{2}}{\sqrt{(b+c)(a+b)}}+\frac{4R(c^3+a^3)\cot\frac{C}{2}}{\sqrt{(b+c)(c+a)}}+\frac{4R(a^3+b^3)\cot\frac{A}{2}}{\sqrt{(a+b)(c+a)}}\overset{A-G}{\ge}$$

$$\ge 12R\sqrt[3]{\left(\prod_{cyc}\frac{b^3+c^3}{b+c}\right)\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=12R\sqrt[3]{\left(\prod_{cyc}(b^2-bc+c^2)\right).\frac{s}{r}}\overset{A-G}{\ge}$$

$$12R.\sqrt[3]{(a^2b^2c^2).\frac{s}{r}}=12R.\sqrt[3]{16R^2r^2s^2.\frac{s}{r}}\overset{Euler}{\ge}12Rs.\sqrt[3]{64r^3}\overset{\substack{Euler\\ and\\ Mitrinovic}}{\ge}48r.2r.3\sqrt{3}r$$

$$\Rightarrow LHS\ge 288\sqrt{3}r^3\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$