# ROMANIAN MATHEMATICAL MAGAZINE

**In $\Delta ABC$ the following relationship holds:**

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}\geq\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}\geq\frac{24^4r^{15}}{(9R^3-64r^3)^4}$$

***Proposed by Zaza Mzhavanadze-Georgia***

***Solution by Tapas Das-India***

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}=\sum\frac{r_a^5}{r_b^2}$$

$$\left(\sum\frac{r_a^5}{r_b^2}\right)\left(\sum r_b^3\right)^4\overset{Holder}{\geq}\left(\sum\sqrt[5]{\frac{r_a^5}{r_a^2}.r_b^{12}}\right)^5=$$

$$=\left(\sum\sqrt[5]{\frac{r_a^5}{r_a^2}.r_b^{10}.r_b^2}\right)^5=\left(\sum\sqrt[5]{\left(r_ar_b^2\right)^5}\right)^5=\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5$$

$$\frac{r_a^5}{r_b^2}+\frac{r_b^5}{r_c^2}+\frac{r_c^5}{r_a^2}\geq\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}$$

$$\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5\overset{AM-GM}{\geq}\left(3\sqrt[3]{r_a^3r_b^3r_c^3}\right)^5=(3s^2r)^5\overset{Mitrinovic}{\geq}$$

$$\geq(3\cdot 27\,r^3)^5=3^{20}r^{15}\ (1)$$

$$\sum r_a^3=\left(\sum r_a\right)^3-3\left(\sum r_a\right)\left(\sum r_ar_b\right)+3r_ar_br_c=$$

$$=(4R+r)^3-3(4R+r)s^2+3s^2r=(4R+r)^3-12s^2R\leq$$

$$\overset{Euler\ \&\ Mitrinovic}{\leq}\left(4R+\frac{R}{2}\right)^3-12.27r^2.2r=\frac{3^6R^3}{8}-3^4.8.r^3=\frac{3^4}{8}(9R^3-64r^3)\ (2)$$

$$\frac{\left(r_ar_b^2+r_br_c^2+r_cr_a^2\right)^5}{\left(r_a^3+r_b^3+r_c^3\right)^4}\overset{(1)\&(2)}{\geq}\frac{3^{20}r^{15}}{\left(\frac{3^4}{8}(9R^3-64r^3)\right)^4}=\frac{3^4.8^4.r^{15}}{(9R^3-64r^3)^4}$$

**Equality holds for an equilateral triangle.**