ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{(a^2+b^2)(a^2c^3+b^5)}{b^2c^2(b^2+c^2)+a^2c^2(a^2+c^2)} \geq 6\sqrt{3}r$$

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$$Walter\ Janous\ inequality: a', b', c'\ and\ x, y, z\ positive\ real\ numbers:$$

$$\frac{x}{y+z}(b'+c') + \frac{y}{z+x}(c'+a') + \frac{z}{x+y}(b'+a') \geq \sqrt{3(a'b'+b'c'+c'a')}\ (1)$$

$$Let\ \frac{a^2+b^2}{c^2} = x, \frac{b^2+c^2}{a^2} = y, \frac{a^2+c^2}{b^2} = z$$
$$and\ \frac{a^3}{c^2} = a', \frac{b^3}{a^2} = b', \frac{c^3}{b^2} = c'$$

$$\frac{(a^2+b^2)(a^2c^3+b^5)}{b^2c^2(b^2+c^2)+a^2c^2(a^2+c^2)} = \frac{\frac{a^2+b^2}{c^2}}{\frac{b^2+c^2}{a^2}+\frac{a^2+c^2}{b^2}}\left(\frac{c^3}{b^2}+\frac{b^3}{a^2}\right) = \frac{x}{y+z}(c'+b')\ (2)$$

$$a'b'+b'a'+c'a' \overset{AM-GM}{\geq} 3\left((a'b'c')^2\right)^{\frac{1}{3}} \overset{Carlitz}{\geq} 3.\left(\frac{4F}{\sqrt{3}}\right) \overset{Mitrinovic}{\geq} 3.(4.3r^2) = 36r^2\ (3)$$

$$\sum \frac{(a^2+b^2)(a^2c^3+b^5)}{b^2c^2(b^2+c^2)+a^2c^2(a^2+c^2)} \overset{(2)}{=}$$

$$= \sum \frac{x}{y+z}(c'+b') \overset{(1)}{\geq} \sqrt{3(a'b'+b'c'+c'a')} \overset{(3)}{\geq} 6\sqrt{3}r$$

Equality holds for an equilateral triangle.