ROMANIAN MATHEMATICAL MAGAZINE

JP.559 If $x, y, z > 0$ then:

$$9\sum_{cyc}\left(\frac{2x+y+z}{x^2+2}\right)^2 \le 2\sum_{cyc}(x^2+2)(y^2+2)$$

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Solution by proposers

Lemma:

If $x, y > 0$ then:

$$2(x^2+2)(y^2+2) \ge 3((x+y)^2+2)$$

Proof.

$$2(x^2y^2+2x^2+2y^2+4) \ge 3(x^2+2xy+y^2+2)$$

$$2x^2y^2+4x^2+4y^2+8-3x^2-6xy-3y^2-6 \ge 0$$

$$2x^2y^2-4xy+2+x^2+y^2-2xy \ge 0$$

$$2(xy-1)^2+(x-y)^2 \ge 0$$

Equality holds for $x = y = 1$.

Back to the problem:

$$(x^2+2)^2(y^2+2)(z^2+2) =$$

$$= (x^2+2)(y^2+2)\cdot(x^2+2)(z^2+2) \overset{Lemma}{\ge}$$

$$\ge \frac{3}{2}((x+y)^2+2)\cdot\frac{3}{2}((x+z)^2+2) =$$

$$= \frac{9}{4}\left((x+y)^2+\left(\sqrt{2}\right)^2\right)\left(\left(\sqrt{2}\right)^2+(x+z)^2\right) \ge$$

$$\overset{CBS}{\ge} \frac{9}{4}\left((x+y)\sqrt{2}+\sqrt{2}(x+z)^2\right) = \frac{9}{4}\cdot 2(x+y+x+z)^2 =$$

$$= \frac{9}{2}(2x+y+z)^2$$

$$2(y^2+2)(z^2+2) \ge 9\left(\frac{2x+y+z}{x^2+2}\right)^2$$

$$9\sum_{cyc}\left(\frac{2x+y+z}{x^2+2}\right)^2 \le 2\sum_{cyc}(x^2+2)(y^2+2)$$